Matched-filter acquisition for BOLD fMRI

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A B S T R A C T

We introduce matched-filter fMRI, which improves BOLD (blood oxygen level dependent) sensitivity by variable-density image acquisition tailored to subsequent image smoothing. Image smoothing is an established post-processing technique used in the vast majority of fMRI studies. Here we show that the signal-to-noise ratio of the resulting smoothed data can be substantially increased by acquisition weighting with a weighting function that matches the k-space filter imposed by the smoothing operation. We derive the theoretical SNR advantage of this strategy and propose a practical implementation of 2D echo-planar acquisition matched to common Gaussian smoothing. To reliably perform the involved variable-speed trajectories, concurrent magnetic field monitoring with NMR probes is used. Using this technique, phantom and in vivo measurements confirm reliable SNR improvement in the order of 30% in a “resting-state” condition and prove robust in different regimes of physiological noise. Furthermore, a preliminary task-based visual fMRI experiment equally suggests a consistent BOLD sensitivity increase in terms of statistical sensitivity (average t-value increase of about 35%). In summary, our study suggests that matched-filter acquisition is an effective means of improving BOLD SNR in studies that rely on image smoothing at the post-processing level.

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Introduction

Spatial smoothing of imaging volumes is ubiquitous in fMRI (Carp, 2012; Pol Movesk et al., 2008). Its routine use before statistical analysis aims at improving the sensitivity and interpretability of blood oxygen level dependent (BOLD) contrast in three ways, i.e., from the perspective of (1) signal processing, (2) statistical inference at the single-subject level and (3) group level inference (Friston, 2007).

Firstly, with respect to the signal processing perspective, smoothing the data with a filter that resembles the spatially extended hemodynamic response is considered optimal to detect activation of this particular shape and scale, according to the matched-filter theorem (Worsley et al., 1996a, b). Secondly, regarding single-subject inference, image smoothing facilitates the application of multiple comparison correction using random field theory (Worsley et al., 1996a, b) since it ensures spatial smoothness of the residual error distribution. Thirdly, at the group level, spatial smoothing helps to absorb anatomical variability between subjects.

Image smoothing is commonly performed with a filtering operation in k-space that attenuates signal content at high spatial frequencies. In doing so it alters the effective point spread function (PSF) such as to broaden its main peak and suppress far-range contamination. However, importantly, variable k-space attenuation not only affects the PSF but also the propagation of noise from raw data into smoothed images. The noise content of the raw data undergoes the same k-space weighting such that the relative impact of noise increases towards the center of k-space. As a consequence, to maximize the SNR of the smoothed data, the raw data should be acquired with variable sensitivity by corresponding k-space weighting at the acquisition level. As will be detailed in the theory part, optimal net SNR is achieved by acquisition weighting that exactly matches the eventual smoothing filter. The underlying mathematics correspond closely to the matched-filter rationale (North, 1963) of the smoothing operation. It is important, however, to distinguish the different filter-matching rationales. Aiming to match the hemodynamic response by smoothing is common practice today and serves for the purposes summarized initially. The utility of also matching data acquisition is a consequence of the smoothing strategy.
and serves exclusively for SNR optimization given a chosen smoothing kernel. Importantly, through appropriate image reconstruction including density correction, the matched-filter acquisition does not change the PSF and thus allows for image post-processing that is identical to standard acquisition.

Acquisition weighting has previously been used to improve the sensitivity of MR spectroscopic imaging and non-proton MRI, summarized under the theme of density-weighted phase encoding (Greiser and von Kienlin, 2003; Greiser et al., 2005; Stobbe and Beaulieu, 2008). In this work, we introduce matched-filter acquisition for fMRI with single-shot echo-planar readouts, which is challenging in that it cannot be accomplished merely by altered phase encoding, but requires 2D trajectory design with complex modulation of k-space velocity. Such trajectories are particularly susceptible to common imperfections of gradient systems such as bandwidth limitations and eddy currents. To gauge and address this issue, we incorporate concurrent magnetic field monitoring (Barmet et al., 2008, 2009, 2010) with NMR probes (Barmet et al., 2010; De Zanche et al., 2008), which permits accounting for imperfections in magnetic field evolution at the image reconstruction stage.

The SNR benefit expected from filter matching relies on the incoherence of noise. In particular, the exact form of the matched-filter acquisition rule proposed here refers to the assumption of independent and identically distributed white noise. Thermal noise, which is prevalent in MR, exhibits this property (Johnson, 1928; Nyquist, 1928). For this noise regime, we show analytically that the distribution of acquisition time should indeed exhibit the same weighting in k-space as the target PSF, to achieve maximum SNR. However, BOLD fMRI is also subject to noise related to physiological processes with non-white statistics (Bianciardi et al., 2009; Krüger and Glover, 2001), including inherent neurophysiological fluctuations as well as respiratory and cardiovascular dynamics (Birn et al., 2008; Chang et al., 2009; Dagli et al., 1999; Glover et al., 2000; Shmueli et al., 2007). Therefore, the experimental validation of matched-filter fMRI in this work comprises signal-to-fluctuation-noise ratio (SNFR) measurements of phantom and in vivo time series, in which we vary the degree of signal-mediated fluctuations and evaluate their influence on the observed SNFR gain. Finally, we perform a proof-of-principle experiment showing the feasibility of matched-filter acquisition also for task-based fMRI. Using a visual paradigm in a preliminary group of four subjects, robust t-value increases are reported over standard EPI acquisition.

Theory and methods

Theory: Matched-densituy acquisition for image post-processing filters

In the following section, we establish the relationship between variable acquisition speed in k-space, the point-spread function (PSF) and signal-to-noise ratio (SNR) within an MR image. This is contrasted to shaping the PSF by retrospective smoothing only.

To estimate SNR, we consider the different propagation of signal and noise in both stages, focusing on the total thermal noise contribution to shaping the PSF by retrospective smoothing only.

\[ \sigma_{\text{final}}^2 = \int_{V_s} \sigma_{\text{final}}^2(k) \, dk. \]  

The optimization constraint is given by a constant total acquisition time \( T_{\text{acq}} \), such that the full optimization problem incorporating relation (3) reads

\[ \int_{V_s} \frac{d_{\text{target}}(k)}{d_{\text{acq}}(k)} \, dk \rightarrow \min \text{ with } \int_{V_s} d_{\text{acq}}(k) \, dk = T_{\text{acq}}. \]  

(5)
The solution to this optimization uses a Lagrange multiplier $\lambda$ and fulfills

$$\frac{\partial |\sigma_{\text{final}}|^2}{\partial (d_{\text{acq}}(k))} = \lambda \frac{\partial}{\partial (d_{\text{acq}}(k))} \int d_{\text{acq}}(k) \, dk - \frac{d_{\text{target}}^2(k)}{d_{\text{acq}}^2(k)} = \lambda \cdot \text{const.}$$  \tag{6}

$$\implies d_{\text{acq}}(k) \propto d_{\text{target}}(k)$$

Therefore, for optimal SNR, target density and acquisition density should be equal in k-space (up to a proportionality factor), in analogy to the matched-filter theorem (Fig. 1; North, 1963). Specifically, in the common case where the target PSF is a Dirac function that maps the object onto image pixels identically, uniform sampling is optimal (Pipe and Duerk, 1995).

The SNR ratio between a matched acquisition and the standard uniform sampling can be expressed via the ratio of final noise variances, i.e.,

$$\frac{\text{SNR}_{\text{matched}}}{\text{SNR}_{\text{final}}} = \left[ \frac{|\sigma_{\text{final}}^\text{uni}|^2}{\sigma_{\text{final}}^2} \right] / \left[ \frac{|\sigma_{\text{final}}|_2^2}{|\sigma_{\text{final}}^\text{matched}|_2^2} \right] \tag{7}$$

We now simplify the numerator and denominator of the right hand side term separately using Eq. (5). For the square of the numerator, the acquisition density fulfills $\int d_{\text{acq}}(k) \, dk = T_{\text{acq}}$, while the uniform sampling density reads $d_{\text{uni}}(k) = T_{\text{acq}}/V_k = \text{const.}$, yielding

$$|\sigma_{\text{final}}^\text{uni}|_2^2 = \int \frac{d_{\text{target}}^2(k)}{V_k} \, dk = \frac{V_k}{T_{\text{acq}}} \int \frac{d_{\text{target}}^2(k)}{V_k} \, dk \quad \tag{8}$$

We normalize the matched acquisition density to $d_{\text{matched}}(k) = C^{-1} \cdot T_{\text{acq}} \cdot d_{\text{target}}(k)$ with $C := \int d_{\text{target}}(k) \, dk$, and, hence, can rewrite the square of the denominator of Eq. (7):

$$|\sigma_{\text{final}}^\text{matched}|_2^2 = \int \frac{d_{\text{target}}^2(k)}{V_k} \, dk = \frac{1}{C} \cdot \frac{1}{T_{\text{acq}}} \int d_{\text{target}}(k) \, dk \quad \tag{9}$$

The SNR ratio between matched and uniform acquisition then evaluates to

$$\frac{\text{SNR}_{\text{matched}}}{\text{SNR}_{\text{final}}} = \sqrt{\frac{1}{\int \frac{d_{\text{target}}^2(k)}{V_k} \, dk}} = \sqrt{C^{-2} \cdot \frac{V_k}{T_{\text{acq}}} \int \frac{d_{\text{target}}^2(k)}{V_k} \, dk} \quad \tag{10}$$

and is therefore proportional to the 2-norm of the target density.

For the fMRI application under consideration, the SNR gain can be explicitly calculated given the ratio between the nominal resolution before smoothing and the FWHM of the Gaussian target PSF, as derived formally in Appendix A. The result is an intrinsic SNR gain capturing the situation of ideal density weighting, which, in practice, is limited by the fidelity of the gradient system. Typical choices of the

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**Post-Processing**

**SNR optimal:**

$$d_{\text{acq}}(k) = \text{FT}(K_{\text{smooth}}(r))$$

$$\sigma_{\text{noise}}^2 \propto \frac{d_{\text{target}}(k)^2}{d_{\text{acq}}(k)^2}$$

Fig. 1. The concept of matched-filter acquisition: After identifying the target post-processing filter $K_{\text{smooth}}$ of a statistical analysis, a corresponding k-space trajectory is designed for image acquisition. Following the matched-filter theorem, this trajectory delivers SNR optimal images, if its variable k-space acquisition density $d_{\text{acq}}$ is equivalent to the Fourier transform of the target post-processing filter.
Gaussian FWHM deliver an intrinsic SNR gain of 2D matched-filter compared to uniform acquisition between 6% and 200% (see Table 1) that depends approximately linearly on the FWHM of the Gaussian kernel.

Intuitively, this linear dependence of SNR gain on FWHM can be understood as follows: Acquisition time is spent inefficiently by uniform sampling in areas of k-space where the target density weighting is low, because noise is accumulated, but the signal information is hardly used. This area shrinks quadratically with the FWHM of the target k-space density and therefore increases quadratically with the FWHM of the target PSF. Thus, the additional standard deviation in the final image increases linearly with this FWHM of the target PSF due to uniform sampling.

Table 1

<table>
<thead>
<tr>
<th>FWHM of target PSF in nominal pixels</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>3.5</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>SNR gain (2D × Intrinsic)</td>
<td>6%</td>
<td>23%</td>
<td>53%</td>
<td>88%</td>
<td>126%</td>
<td>163%</td>
<td>201%</td>
</tr>
<tr>
<td>SNR gain (2D × Reference Experiment)</td>
<td>3%</td>
<td>12%</td>
<td>25%</td>
<td>40%</td>
<td>55%</td>
<td>72%</td>
<td>92%</td>
</tr>
</tbody>
</table>

Theory: EPI trajectory design for a matched-filter Gaussian density

As a k-space trajectory is needed for MR image encoding, we now show how to transform the target acquisition density obtained in the last section into a continuous trajectory subject to gradient system limitations. Specifically, we design a single-shot Gaussian acquisition density 2D EPI trajectory to enable a direct comparison with the most prominent fMRI acquisition technique, i.e., a 2D EPI with uniform acquisition density.

In principle, the k-space acquisition density can be altered by two aspects of trajectory specification: Firstly, the shape of the trajectory can be modified by variable spacing of different trajectory segments, such as EPI traverses or spiral revolutions (Greiser and von Kienlin, 2003; Greiser et al., 2005; Kim et al., 2003; Spielman et al., 1995). Secondly, the velocity along the trajectory can be modulated, thus distributing acquisition time variably in k-space.

For EPI, modifying the trajectory shape could be readily implemented by only modulating the density of traverses. A variable time allocation would be accomplished, as more acquisition time is allocated to regions with relatively more traverses. On the downside, too much acquisition time is deployed compared to the Nyquist sampling density needed. To minimize the number of such turns, more traverses would accrue more traverse turns locally, such that the time spent in this particular part of k-space and therefore increases quadratically with the FWHM of the target PSF. Thus, the additional standard deviation in the final image increases linearly with this FWHM of the target PSF due to uniform sampling.

For traverses where $C_1$ exceeds the maximum gradient amplitude, the Gaussian acquisition density is replaced by a baseline uniform density ensuring Nyquist sampling.

In practice, due to gradient amplitude and slew rate limitations, this time course cannot be realized at the traverse turns which require the fastest possible gradient switching.

To achieve a Gaussian density weighting along the phase encoding direction, $k_y$, we vary the acquisition time spent on different traverses, $T_{traverse}$. While the gradient shape follows Eq. (12) for all traverses, the scaling factor $C_1$ of the gradient amplitude depends on the $k_y$-coordinate of the specific traverse, according to Eqs. (6) and (12):

$$C_1 = \frac{1}{T_{traverse}(k_y)} \frac{1}{d_{target}(k_y)} \exp \left(-\frac{k_y^2\omega^2}{2}\right).$$

For traverses where $C_1$ exceeds the maximum gradient amplitude, the Gaussian acquisition density is replaced by a baseline uniform density ensuring Nyquist sampling.

For all our experiments, data were acquired on a Philips 3 T Achieva (Best, The Netherlands) system equipped with an 8-channel head coil (Philips, Best, The Netherlands) and gradient specifications of 31 mT/m maximum amplitude and 200 T/m/s maximum slew rate. We compared a matched-filter EPI to a uniform EPI trajectory that shared the following acquisition parameters: TR 3 s (phantom data: 6.25 s), TE 35 ms, readout duration 41 ms, receiver bandwidth 375 kHz, FOV 227 mm, SENSE reduction factor 3, voxel size 1.8 x 1.8 x 3 mm$^3$, 5 slices with 3 mm between-slice gap. The target filter was a 2D Gaussian with a FWHM of 4.5 mm (or 2.5 voxels).

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1 According to a recent meta-study on 300 fMRI studies (Carp, 2012), 88% of all studies reported using (presumably Gaussian) smoothing, and nearly more than half of them with a FWHM of 8 mm and larger, which corresponds to more than 2.5 pixels for voxel sizes of 3 mm and below.
We deliberately designed the uniform EPI to have the same readout duration as the matched-filter EPI (41 ms), although the gradient specifications would have allowed for a faster uniform readout (29 ms), and hence, more volumes per time. This prolonged readout enabled a fair SFNR comparison to the matched-filter trajectory, because the increase in temporal SNR when decreasing slice TR \( \approx TE + T_{2\text{co}}/2 \) from 56 to 50 ms is outweighed by the SNR loss per image when decreasing image acquisition time from 41 to 29 ms.

Separate \( B_0 \) and coil receive sensitivity maps were acquired in each session (TR 800 ms, TE 1 ms, \( \Delta T = 2.3 \) ms, spin-warp images with a resolution of \( 1 \times 1 \times 3 \) mm\(^3\)).

Our implementation of the matched-filter Gaussian EPI trajectory using variable gradient strengths is particularly susceptible to any kind of gradient imperfections. There are two reasons for this sensitivity: On the one hand, the gradient system is operated close to its limits to enable a large range of velocity modulation. On the other hand, the non-periodic gradient time course precludes standard correction methods for gradient inaccuracies, such as EPI phase correction. Therefore, we utilized concurrent magnetic field monitoring as introduced by Barmet et al. (2008, 2009, 2010, 2011) to (1) investigate the feasibility and accuracy of the demanding gradient evolutions of the proposed matched-filter EPIs on a clinical MR system, (2) study deviations in \( k \)-trajectories and \( k \)-space densities from the ideal matched-filter EPI and (3) enable image reconstructions informed by the actual, measured \( k \)-space trajectory. In the hetero-nuclear monitoring setup 16 transmit/receive \(^1\)H NMR field probes were attached to the head coil (Barmet et al., 2010; De Zanche et al., 2008; Wilm et al., 2011). The acquired probe phase evolutions were expanded into real-valued spherical harmonics (Barmet et al., 2008; Vannesjö et al., 2013b; Wilm et al., 2011), yielding phase coefficients for the global phase, \( k_g(t) \), the linear \( k \)-space, \( k_{l}(t) \), \( k_{s}(t) \) and \( k_{c}(t) \) and second order phase coefficients \( k_{4,6}(t) \) over the entire readout.

All image reconstructions, for both matched-filter and uniform EPI, were performed using concurrent field monitoring data from the probe phase fits. Global \( k_0 \) phase information was used for demodulation of the raw coil data of the 8-channel head coil. Afterwards, the images were reconstructed from the demodulated data in combination with \( 1^{\text{st}} \) order \( k \)-space trajectory information \( k_{x}, k_{y}, k_{z} \) in an iterative, gridding-based, conjugate-gradient SENSE algorithm (Beatty et al., 2005; Jackson et al., 1991; Pruessmann et al., 2001), using an in-house Matlab implementation (The MathWorks, Natick, MA). This algorithm was augmented with multi-frequency interpolation (MFI) for static \( B_0 \)-field correction (Man et al., 1997; Sutton et al., 2003). The application of this reconstruction algorithm ensured an SNR optimal image reconstruction where the target PSF was achieved post hoc via smoothing, which is equivalent to direct reconstruction with the target PSF (Pruessmann and Taso, 2008). In particular, the objective function of the reconstruction imposed a Dirac target PSF, thus permitting an implicit density correction for the variable acquisition density of the matched-filter trajectory. Consequently, both matched-filter and uniform EPI scans exhibited the same image resolution and smoothness before entering statistical pre-processing.

**Experiment 1: Assessment of SFNR gain for EPI time series in different noise regimes**

The first experiment assessed the validity of the matched-filter acquisition argument for different noise regimes with different levels of signal fluctuations. Phantom data were acquired from a water-filled sphere. In vivo data were acquired from 4 healthy volunteers (1 male) after written informed consent and with approval of the local ethics committee. Subjects were asked to lie still in the scanner with their eyes closed (i.e., a “resting-state” condition).

For both uniform and matched-filter EPI, we acquired 9 sessions with different excitation flip angles (0, 5, 15, 25, 40, 50, 60, 75 and 90 degrees) to vary the signal content and therefore the contribution of signal-dependent noise. Each session contained 95 scans for the phantom data, and 48 scans for in vivo sessions, plus 5 void scans to minimize saturation effects. Realignment and smoothing with the matched target kernel were performed on all images using SPM8 (http://www.fil.ion.ucl.ac.uk/spm/).

We considered SFNR ROI-wise by first determining the mean of the magnitude signal within the ROI for each image. Then, the SFNR was computed as the ratio of the temporal mean of this ROI mean and its temporal standard deviation. The ROI in the phantom was a centered disc in each slice extending to 2/3 of the object diameter to avoid edge effects. For the in vivo case, subject-specific gray matter, white matter and CSF regions—that suffer from different levels of physiological noise (Krüger and Glover, 2001; Triantafyllou et al., 2006)—were included as separate ROIs in the analysis. These regions were extracted using a bias field correction (Salvado et al., 2006) and a k-means clustering algorithm on the spin-warp TE, image.

**Experiment 2: fMRI paradigm and analysis**

In the second experiment, the benefits of matched-filter EPI acquisition were assessed in a visual fMRI paradigm using t-contrast values as a summary statistic of activation detection.

The fMRI paradigm was designed to stimulate the quarter-fields of the visual cortex: 16 s of flickering, color-changing wedges were interleaved with 5 s of fixation; 8 blocks of upper left/lower right (ULLR) and upper right/lower left (URLL) wedges were presented over 120 scans (TR 3 s). The visual presentation was performed using a projector (resolution 800 × 600) and a mirror mounted on the head coil. Subjects’ attention was maintained using a simple button response task to any contrast alteration of the fixation point.

The data were acquired in the same subjects and—apart from subject 4—on the same measurement day as for experiment 1. Two sessions of each uniform EPI and matched-filter EPI acquisition were measured to compare within-modality variance to between-modality variance of the statistical results. The order of matched and uniform acquisitions was counterbalanced between subjects, and independently for the 1st and 2nd repetition of these sessions. The slice geometry was equivalent to experiment 1. Specifically, slice orientation was oblique transverse, parallel to the calcarine sulcus to cover visual cortex. Peripheral physiological measures characterizing cardiac pulsation and the respiratory cycle were recorded simultaneously with fMRI using an electrocardiogram (ECG) and breathing belt, respectively.

Spatial preprocessing and statistical analysis of the fMRI data were performed in SPM8. Pre-processing included realignment and spatial smoothing with the target Gaussian PSF. The general linear model (GLM) for the statistical analysis included a canonical hemodynamic response function (HRF) and temporal/ dispersion derivative regressors of the ULLR and URLL blocks. Furthermore, we included 2 types of nuisance regressors into the GLM: 6 movement parameters from realignment and physiological noise modeling using RETROICOR (Glover et al., 2000). Our specific implementation of RETROICOR, the physIO Toolbox (Kasper et al., 2009; open source code available as part of the TAPAS software collection: http://www.translationalneuromodeling.org/tapas/) uses Fourier expansions of different order for the estimated phases of cardiac pulsation (3rd order), respiration (4th order) and cardiorespiratory interactions (1st order) following (Harvey et al., 2008).

The statistical results were assessed on t-maps contrasting ULLR–URLL (contrast 1) and URLL–ULLR (contrast 2). Both peak t-value and total cluster sizes were compared between matched and uniform acquisition sessions. All results were \( p = 0.05 \) FWE-peak level corrected for the whole acquisition volume. Finally, for a more quantitative handle on BOLD sensitivity, we performed a total least squares regression (TLS, modified Matlab implementation; Hall, 2011) of all corresponding brain voxels for the t-value change of one session compared to a reference session. TLS reports an average t-value change over all voxels,
and thus a more robust summary measure than peak t-values or activation extent.

Results

Monitoring: Trajectories and k-space densities

For the implemented matched-filter EPI, the concurrently monitored encoding magnetic fields, phase coefficients and corresponding k-space densities are shown in Figs. 2–4, respectively. We present the measured matched-filter readout for the representative 5th scan of the 1st fMRI session of subject 3, but the asserted statements hold for all observed readouts.

In general, the demanding non-trapezoidal readout gradient waveform is reproduced quite accurately by the gradient system, exhibiting only the common bandwidth limitation of the gradients, smoothing of switching events and small gradient delays (Fig. 2, red curve, compared to black curve of nominal gradient evolution). Note that the amplitude of the measured phase encoding gradient blip is also greatly reduced due to this low-pass filter property of the gradient chain (Fig. 2B), but its area is preserved due to commensurate broadening of the blip (cf. the EPI traverse spacing in Fig. 4A.)

Fig. 3 shows the phase evolutions induced by this gradient waveform expanded in 0th to 2nd spatial order spherical harmonics: for the monitored global phase $k_0$, a roughly linear increase during the readout is evident that carries the distinct sinusoidal modulation of the EPI traverses (about 300 Hz). This modulation presumably stems from slight $B_0$ eddy currents induced by the readout gradient. Similarly, the reduced slope of the linear component of $k_0$ during the central part of the readout might result from the lower frequency of phase encoding gradients and their concomitant $B_0$ eddy currents for the inner, density-weighted traverses.

The linear phase coefficients (Fig. 3, middle panel), i.e. the k-space representation of the trajectory, exhibit two main deviations from the nominal matched-filter EPI, which are best visible in the classical 2D representation of the trajectory (Fig. 4): Firstly, we found a compression of about 25 rad/m of the trajectory in frequency encoding direction, resulting in a slightly reduced actual image resolution, which has also been reported for uniform EPI trajectories (Vannesjo et al., 2013b). Secondly, the actual sampling points within the traverse did not coincide exactly with the nominal positions but deviated by up to one Nyquist sampling interval in frequency encoding direction (Fig. 4, zoomed panels).

In turn, the distribution of these actual sampling points determines the realized acquisition density $d_{acq}$, whose resemblance to a Gaussian
is crucial for the expected SNR gains derived in the theory section of this paper. Visually, the 2D sampling point distribution (Fig. 4) indicates a density weighting with rotational symmetry which was assessed quantitatively using a gridding-based estimation of the acquisition density from the sampling points (Jackson et al., 1991). Indeed, the acquisition density is Gaussian (Fig. 4B), but, compared to the nominal density, exhibits a slight reduction (root mean square error, RMSE, 3%) in k-space center, i.e. for $|k| < 60\% k_{max}$ and considerable overshoot (RMSE 20%) at its periphery, i.e. the EPI turns (Fig. 4C).

In summary, even though the individual positions of the k-space samples vary between nominal and actual trajectory, the induced densities exhibit high similarity and render the matched-filter prerequisites on the expected SNR gains valid.

Monitoring: Image reconstruction

Fig. 5 shows the unsmoothed reconstructed images of the undersampled, single-shot variable-density EPI acquisition in comparison to the spin-warp image acquired for coil sensitivity estimation. The matched-filter EPI (Fig. 5A) exhibits a low level of artifacts and high geometric congruency to the spin-warp image used as anatomical reference (Fig. 5C). Specifically, the edges of the brain, CSF and gray/white matter boundaries coincide in the matched-filter EPI and the anatomical reference (Fig. 5B, edges of spin-warp image overlayed on matched-filter EPI).

We investigated the particular impact of concurrent field monitoring on image quality in a series of alternative reconstructions, where we either used the nominal trajectory, the fully monitored 1st order trajectory

![Figure 4. Measured 2D sampling scheme and k-space acquisition densities of the matched-filter EPI trajectory.](image-url)
including the global phase $k_0$ or a hybrid reconstruction with measured global phase, but nominal $k_x$ and $k_y$, as input to the gridding-based iterative reconstruction (Fig. 6). The resulting images show the necessity of a reconstruction utilizing full knowledge about the actual trajectory and global phase. While this image reconstruction is virtually artifact-free, a reconstruction on the sole nominal trajectory exhibits both ghosting and blurring artifacts (Fig. 6B, D, G, I). The reconstruction incorporating the measured global phase to the nominal trajectory sheds light on the different artifact mechanisms (Fig. 6C, E, H, J): The ghosting edges parallel to phase encoding direction $k_y$ are greatly reduced for this reconstruction, hence they mainly stem from a mismatch in $k_y$ during the readout. However, the blurred, rippled edges along frequency encoding direction remain and, thus, are presumably related to the gradient impulse response-induced compression of the matched-filter trajectory along $k_x$.

### SFNR analysis

We evaluated local SFNR for each EPI session as a function of signal strength, which is proportional to the sine of the excitation flip angle (Fig. 7). First, we captured the statistics of pure thermal noise by measuring a session with $0^\circ$ excitation flip angle, both in the phantom and in vivo (Fig. 7, horizontal dashed lines). Before reconstruction, each of these measured noise instances was added to the coil data of one fixed scan of the 90° session to evaluate pure thermal noise influence on SFNR. In this limiting case, the local SFNR gain of matched-filter compared to uniform EPI reached 45%, in good congruence with the theoretical expectation of 41%.

Secondly, in the phantom, the SFNR increased with signal level for both, matched-filter and uniform EPI, but with a steeper slope for matched-filter EPI, thus preserving an SFNR advantage compared to uniform EPI. However, the observed SFNR gain decreased for higher signal level (Fig. 7A, blue-shaded area). This indicates MR signal fluctuations in addition to thermal noise, which could occur at any stage of the excitation, encoding and reception process. Nevertheless, even for the practically relevant case of high signal level, the SFNR gain in the phantom remained well above 30% for matched-filter compared to uniform EPI.

Finally, for the in vivo measurements, we found regional differences in the SFNR dependence on signal level, most likely due to varying contributions of physiological noise in the areas considered: For white matter ROIs, the SFNR curves resembled those in the phantom, but exhibiting a lower minimal SFNR gain at high signal levels of about 20% for matched-filter compared to uniform EPI (Fig. 7B). In areas containing cerebro-spinal fluid (CSF), on the other hand, SFNR increased at low, but decreased at high signal level, presumably due to the strong pulsatile physiological noise (Fig. 7D; Krüger and Glover, 2001). However, we could still observe a relative gain in SFNR of about 15% for matched-filter compared to uniform EPI. For fMRI-relevant gray matter ROIs, an SFNR gain of up to 20% was found at high signal levels. The individual matched and uniform SFNR curves resembled those in white matter qualitatively, while the SFNR ratio exhibited decay with flip angle as in CSF, presumably reflecting the intermediate contribution of physiological noise in gray matter regions compared to white matter and CSF (Fig. 7C).

### fMRI analysis: t-maps and total least squares

In the individual SPM analysis of each acquired session, all contrast maps showed the expected activations patterns, representing the quarter-fields in the visual cortex by contrasting the two stimulation blocks either as ULLR–URLR (contrast 1, Fig. 8, red voxels) or URLR–ULLR (contrast 2, Fig. 8, green voxels). The activation patterns are visualized as overlays on an EPI scan of the corresponding session and show nice alignment with gray matter structures in the individual subject. Comparing the t-maps in terms of peak t-values and cluster sizes of significant voxels, the sessions with matched-filter acquisitions outperformed uniform EPI acquisitions consistently within subjects (across sessions) and between subjects (Fig. 8, Table 2). In particular,
for both the first and second repetition (effect of session) of each acquisition in all subjects, matched-filter EPI provided superior activation patterns (according to the aforementioned criteria) than uniform EPI, whereas the activation patterns within an acquisition modality resembled each other in the 1st and 2nd repetition.

For a more quantitative and comprehensive view on this improved BOLD contrast sensitivity, especially its robustness and test/retest reliability, we generated a scatter plot depicting each individual voxel for both the 1st and 2nd repetition (effect of session) of each acquisition in all subjects, matched-filter EPI provided superior activation patterns (according to the aforementioned criteria) than uniform EPI, whereas the activation patterns within an acquisition modality resembled each other in the 1st and 2nd repetition.

This data representation was evaluated using a total least squares (TLS) estimation of the mean slope $\Delta t / t$, which indicates the relative increase in contrast (and therefore BOLD) sensitivity for the session of interest compared to the reference session, averaged over all voxels significant in both sessions (Table 3). TLS is an extension of ordinary least squares regression for cases where both dependent and independent variables contain observation noise, as in our case.

At the single-subject level, performing TLS between matched-filter and uniform EPI sessions assessed the average effect of the acquisition scheme on BOLD sensitivity (Fig. 9A, B), while TLS between session 1 and 2 of the same acquisition scheme (uniform or matched) provided a measure of test–retest reliability (Fig. 9C, D), with a horizontal line indicating identical replication. For example, in the most consistent data set (subject 4, Fig. 9A–D), the TLS analyses comparing matched and uniform acquisition yielded a positive slope indicating a $t$-value increase of $35\% \pm 2\%$ (95% confidence limits using bootstrapping) in session 1 (Fig. 9A) and $41\% \pm 2\%$ in session 2 (Fig. 9B). At the same time, the TLS analyses comparing sessions 1 and 2 found high test–retest reliability within each acquisition scheme, with session differences of only
The average effect of the acquisition scheme on BOLD sensitivity was consistently found in all other subjects as well: TLS yielded exclusively positive slopes when comparing matched to uniform sessions (Table 3), with the average t-value gain ranging from 14% to 146%. The magnitude of this t-value gain typically corresponded to the increase in cluster extent and/or peak t-value of significant voxels (Table 2). However, two subjects (1 and 3) exhibited poor test–retest reliability in TLS analyses between sessions 1 and 2 within each acquisition scheme, presumably due to bulk motion and subsequent voxel mis-registration, that might explain the unexpectedly high sensitivity gains of 83% ± 3% and 146% ± 19%. Nevertheless, even for these subjects, matched-filter acquisition consistently showed higher peak level.
and/or larger cluster extents compared to the uniform session measured back-to-back, i.e. matched 1 vs. uniform 1 and matched 2 vs. uniform 2 (Table 2).

At the group level, a pooled TLS analysis comprising all significant voxels of all subjects and sessions yielded an average \( t \)-value gain of 37% for matched-filter compared to uniform EPI acquisitions of corresponding sessions (Fig. 9E). This main effect of acquisition scheme was also significant in a two-way (acquisition by session) repeated measures analysis of variance (ANOVA) of the TLS slopes (Table 3) normalized to uniform session 1 (\( F(1,3) = 14.0, p < 0.05 \)). No main effect of session or interaction between acquisition and session could be found (\( F(1,3) = 0.37, p > 0.58; F(1,3) = 0.46, p > 0.54 \)). Due to the small number of measurements, we also performed a non-parametric test on the TLS slopes that confirmed the main effect of acquisition, with matched > uniform directionality (Mann–Whitney \( U = 58, n_1 = n_2 = 8, p < 0.005 \) one-tailed; implementation: http://vassarstats.net).

Discussion and conclusion

The results presented have shown the feasibility of a matched-filter acquisition for fMRI in four stages: First, we verified that a single-shot EPI trajectory with Gaussian acquisition density and typical resolution and readout duration can be accomplished within the limits of a commercial MR gradient system. The concurrent field monitoring results confirmed that the experimentally realized acquisition density was indeed Gaussian, as is optimal for the Gaussian filter post-processing, with a root mean squared error (compared to the prescribed Gaussian density) of 3% in k-space center and 20% at the EPI turns.

Secondly, virtually artifact-free image reconstructions could be retrieved from these matched-filter k-space trajectories. To this end, it was crucial to perform image reconstructions using both static and dynamic field monitoring for highest image quality. Concurrent field monitoring proved to be a robust and reliable method for correcting eddy-current and gradient imperfections. In this study, effects on \( B_0 \) and \( 1^\text{st} \) order phase coefficients entered our image reconstruction (although extensions to incorporate concurrently monitored higher-order phase information exist; Wilin et al., 2011, 2012). Alternatively, reproducible deviations from the nominal k-space trajectory could be corrected using a calibration-based image reconstruction method (Graedel et al., 2013) that relies on the characterization of the gradient impulse response function in an independent, field monitoring-based experiment (Vannegos et al., 2013a,b).

Thirdly, we have seen that the experimentally obtained SFNR improvements in the phantom and in vivo match the theoretically derived SFNR gains of 40% very well in the regime of thermal noise. Furthermore, even in the regime of high signal-induced noise contributions, a substantial SFNR increase of about 20% could be retained. Last, and most importantly, these SFNR increases translated into improved sensitivity for task-based fMRI contrasts, as demonstrated...
by a comparison of voxel-wise t-statistics under matched-filter and uniform EPI acquisition. A total least squares analysis for t-values of corresponding voxels confirmed that t-statistics of significant voxels were reproducibly higher for matched-filter fMRI by 20–40%. On top of that, again using TLS, we confirmed that in most subjects this difference was considerably higher than the within-modality t-value fluctuations of matched-filter and uniform EPI acquisition sessions, though the small number of subjects precludes a generalization of these preliminary findings.

In summary, it is remarkable that despite the multiple potential image artifact mechanisms and limited scope of our theoretical noise considerations, a considerable portion of the theoretical SNR advantage of matched-filter fMRI could be preserved through these four stages. Still, the quantitative progressions of the realized SFNR gains within and through these stages deserves further discussion here; most prominently (1) the decrease of the in vivo SFNR gain from 40% to 20% for high signal levels, and (2) the subsequent rise of contrast-to-noise ratio (CNR) advantage in task-based fMRI to 20–40% compared to the aforementioned 20% SFNR increase in resting-state. Both observations arise from the intricate noise situation for in vivo time series, which violates the white Gaussian noise assumption exploited in our theoretical treatment of matched-filter acquisitions. These deviations from a flat noise spectrum are induced by fluctuations in the measured MR signal and can be categorized into two classes: system-dependent and object-dependent fluctuations.

Typically, object-dependent fluctuations are the dominant non-white noise source in MRI, particularly at high main field strength (Krüger and Glover, 2001; Triantafyllou et al., 2006). They arise from the measured physiological systems themselves, which frequently exhibit a tendency towards low frequency noise, i.e. a “pink” noise spectrum, e.g. through breathing and cardiac pulsation (Birn et al., 2008; Chang et al., 2009; Dagli et al., 1999; Glover et al., 2000; Shmueli et al., 2007). System-dependent MR signal fluctuations, on the other hand, were particularly small in our measurements, since the concurrent field-monitoring approach corrected for instabilities in the encoding main and gradient fields, as well as any clock jitter of the spectrometer. Still, non-white noise components might have been introduced in the transmission and reception chain of the system, i.e. the excitation B1 field and receiver gain, respectively.

A general quantification of these system- and object-dependent noise spectra is challenging, but may in principle serve two applications. Firstly, the prediction of in vivo matched-filter SFNR gains could become more accurate when deduced from a pink noise spectrum. Specifically, the upper bound on SFNR gain computed in the theory section for white noise would become tighter, because acquisition weighting unfolds its full strength if noise adds up incoherently, i.e. for white noise. In principle, these pink noise assumptions could then predict the aforementioned deviation between low and high signal level SFNR gain in our resting-state experiments.

Secondly, one could envisage a pink noise spectrum dictating different acquisition strategies to achieve the maximum SFNR gain. However, pink noise shares spectral characteristics with the BOLD signal of interest, thus accruing with similar coherence over time. Consequently, both pink noise and BOLD signal might be suppressed by an acquisition matched in this way, causing information loss. For pink noise, it therefore seems preferable to rely not only on noise statistics, but rather the exact knowledge of the occurring noise instances. Actual instances of physiological noise, for example, can be readily modeled from peripheral measures, such as ECG and breathing belts, and used as confound regressors to de-noise voxel time series, e.g. using RETROICOR (Glover et al., 2000; Hutton et al., 2011; Kasper et al., 2009). The second quantitative deviation in our experiments (up to 40% CNR increase for task-based fMRI compared to only 20% SFNR increase in “resting-state” fMRI) may be understood as a special case of this pink noise correction: While spontaneous BOLD fluctuations in resting-state were considered “noise”, thus lowering the SFNR, the contrast-related BOLD responses in significant voxels were identified as signal of interest and therefore did not contribute to the residual error, i.e. noise amplitude, for the task-based fMRI sessions. Consequently, also resting-state connectivity analysis (Biswal et al., 1995), in contrast to pure SFNR measurements, should benefit from matched-filter acquisition on the same order as task-based fMRI, because correlations in (BOLD) signal fluctuations become a signal of interest here. Hence, the confounding noise in correlation detection has a noise spectrum more similar to white noise and the matched-filter theory assumptions.

In this work, we exemplified the principle of matched-filter acquisition, showing how a variable density 2D EPI readout can be used in fMRI to achieve SNR optimality for a Gaussian target PSF. The general framework of matched-filter acquisition, however, is not restricted to any of the four design decisions made here; neither the EPI readout, nor the Gaussian target PSF, nor the fMRI application and not even the criterion of SNR optimality. In the following, we will conclude with an outlook to possible extensions of matched-filter acquisitions regarding these four aspects, in order of increasing generality:

Choosing a 2D EPI trajectory to implement a variable density acquisition was motivated by demonstrating the matched-filter principle for the currently most robust and commonly used readout in fMRI. However, as shown in Eq. (10), the SFNR gain scales with the square root of covered k-space volume, which promises an even greater advantage for 3D matched-filter acquisitions, such as concentric shell trajectories (Zahnseisen et al., 2012) or the 3D EPIs recently adopted for fMRI (Lutti et al., 2013; Poser et al., 2010). More generally, the choice of the EPI trajectory itself for a Gaussian smoothing kernel is suboptimal. Inevitably, the EPI turns at traverse ends waste acquisition time in the de-emphasized high-frequency regime of the target PSF. Spiral readouts (Ahn et al., 1986; Glover and Lai, 1998) might be natural alternatives to implement a Gaussian acquisition density, since they are rotationally symmetric, and have been successfully utilized for variable density acquisitions before (Chang and Glover, 2011). Moreover, they feature only a few sharp turns in the k-space center, thus allowing for more efficiency in realizing the prescribed acquisition density. However, their sensitivity to static B0 inhomogeneity poses a considerable challenge (Bornert et al., 1999).

The next design consideration refers to the selection of a Gaussian target PSF itself. While smoothing with a Gaussian kernel is prevalent, other filters for image post-processing in fMRI applications have been proposed, such as prolate spheroidal functions or wavelets (Lindquist

### Table 3

BOLD sensitivity increase in task-based fMRI using matched-filter acquisition. The mean t-value increase (including 95% confidence limits determined by bootstrapping) is reported, as computed by the TLS analysis for the relevant contrasts, comparing all pairs of sessions (columns) for all subjects (rows). A graphical representation of the corresponding data is depicted in Fig. 9, in particular all four pair-wise session comparisons of subject 4 (Fig. 9A–D).

<table>
<thead>
<tr>
<th>Subject</th>
<th>Mean percent t-value change and 95% confidence limits</th>
<th>Uniform session 2 vs. session 1</th>
<th>Matched session 2 vs. session 1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Session 1 matched vs. uniform</td>
<td>Session 2 matched vs. uniform</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>39.7 (7.4)</td>
<td>145.6 (18.9)</td>
<td>−462.2 (21.1)</td>
</tr>
<tr>
<td>2</td>
<td>23.7 (13.1)</td>
<td>14.7 (1.5)</td>
<td>119.1 (1.3)</td>
</tr>
<tr>
<td>3</td>
<td>83.1 (2.4)</td>
<td>42.7 (4.2)</td>
<td>80.3 (4.4)</td>
</tr>
<tr>
<td>4</td>
<td>34.8 (1.3)</td>
<td>41.2 (1.4)</td>
<td>−63.1 (1.1)</td>
</tr>
</tbody>
</table>
and Wager, 2008; Van De Ville et al., 2006; Yang et al., 2002). The rationale for matched-filter acquisition holds unaltered for any target PSF, as do the global SNR gains derived in Eq. (10), as long as the target PSF is shift-invariant. For more specialized applications, e.g. cortical surface mapping or adaptive smoothing (Andrade et al., 2001; Harrison et al., 2008; Tabelow et al., 2006), taking into account regional anatomical variability for kernel adaptation, a matched-filter acquisition strategy will achieve SNR optimality for one pre-selected kernel, i.e. only locally in the image.

Beyond fMRI, fast single-shot readouts using matched-filter acquisitions may have applications in other notoriously low-SNR measurements such as echo-planar spectroscopic imaging (EPSI), diffusion- or perfusion-weighted imaging. Here, other target PSFs, such as Hamming filters for ringing suppression, might be desirable (Greiser et al., 2005; Kasper et al., 2012; Stobbe and Beaulieu, 2008). However, enforcing density weighting via gradient modulation for an arbitrary target PSF requires a more general method to design the gradient waveform than the one presented here for the Gaussian PSF. To this end, a promising algorithm for time-optimal gradient waveform design was presented by (Lustig et al., 2008), which could be extended to allow for variable k-space densities through arc-length parameterized gradient limits along the trajectory, hence implementing Eq. (11).

One limitation of our approach to achieve acquisition density weighting with a variable velocity in k-space is the requirement of sufficient flexibility for gradient modulation. Therefore, readouts that require maximum gradient amplitude or slew rate at all times cannot be augmented by a matched-filter acquisition. Objectives like robustness to $T_2^*$ and off-resonance effects, ultra-high spatial resolution or large slice coverage demand these maximally fast readouts. Alternatively, in these cases, a partial matched-filter acquisition weighting can be achieved in phase encoding direction by varying phase blip gradient moments, i.e. spacing of k-space traverses (Kasper et al., 2010; Zeller et al., 2013). Such variable spacing of k-space lines typically violates Nyquist sampling and thus necessitates parallel imaging or additional k-space traverses at the expense of lower acquisition efficiency (due to additional traversal turns). Inherently, it can only achieve filter-matching in phase encoding direction, limiting the expected SNR increase to the square root of the 2D matched-filter acquisition through velocity variation as presented here. Moreover, recent advances in gradient performance that allow for maximum gradient strengths of 100–300 mT/m and slew rates above 200 T/m/s (Kimmel et al., 2012; Van Essen et al., 2012), might lift current constraints on velocity-modulated matched-filter acquisition in the near future to open up its versatility and application range even further.

On a final, conceptual note, SNR optimality is only one, albeit important, criterion for coordinating acquisition and reconstruction to shape signal and noise behavior. According to our Eq. (3), acquisition weighting can be recruited to design any noise variance landscape in k-space for a given target PSF, as the final noise variance in k-space is simply the ratio of the squared target PSF and acquisition density at each k-space position. For example, setting the acquisition density to a multiple of the squared target PSF, the dependence on k between numerator and denominator in Eq. (3) cancels out. Hence, the final noise variance in k-space is constant, i.e. flat, and, by the Fourier autocorrelation theorem, the noise variances in image space are uncorrelated. Taken together, this means that we can achieve voxel-wise noise decorrelation in an MR image by making the acquisition density proportional to the square of the target PSF. In this way, acquisition density matching could broaden its scope to areas where the delineation of unique signal contributions per voxel is crucial, such as high-resolution, layer-specific fMRI (Goense et al., 2012; Koopmans et al., 2010) and multivariate statistical analyses of fMRI data (Haynes and Rees, 2006), or applications that rely on voxel correlation measures, e.g. "resting-state" functional connectivity (Biswal et al., 1995; Buckner et al., 2013; Cole et al., 2010).

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Appendix A

We calculate the expected SNR gain for a Gaussian smoothing kernel with a FWHM defining a k-space target density:

$$d_{\text{target}}(k) = \begin{cases} C \cdot \exp \left( -\frac{k^2 \sigma_r^2}{2} \right) & \text{for } |k| \leq k_{\text{max}} \\ 0 & \text{otherwise} \end{cases}$$

(A.1)

with $k_{\text{max}} = \frac{\pi}{\Delta} \text{ and } \sigma_r = \text{FWHM}/\sqrt{8 \ln 2}$ and $C = \text{const.}$, such that $\int d_{\text{target}}(k) \, dk = 1$.

As the Gaussian kernel is separable, we can calculate the SNR gain for each acquisition dimension individually utilizing Eq. (10), where $V_k = 2 \cdot k_{\text{max}}$, and yield, for a d-dimensional scan:

$$\frac{\text{SNR}_{\text{matched}}}{\text{SNR}_{\text{uni}}} = \left( 2 \cdot k_{\text{max}} \cdot \int_{-k_{\text{max}}}^{k_{\text{max}}} d_{\text{target}}(k) \, dk \right)^{\frac{1}{2}}.$$  

(A.2)

The integral on the right-hand side can be expressed using the error function $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{-x}^{x} \exp(-t^2) \, dt$ via variable substitution $k \rightarrow k \sigma_r$, yielding

$$\frac{\text{SNR}_{\text{matched}}}{\text{SNR}_{\text{uni}}} = \left( 2 \cdot k_{\text{max}} \cdot \sqrt{\frac{8}{\pi}} \cdot \text{erf}(k_{\text{max}} \sigma_r) \right)^{\frac{1}{2}}.$$  

(A.3)

Because of $\int_{-k_{\text{max}}}^{k_{\text{max}}} d_{\text{target}}(k) \, dk = 1$, $C$ itself can be expressed via the error function as $C^{-1} = \sqrt{2 \pi} \text{erf}(k_{\text{max}} \sigma_r / \sqrt{2})$ to arrive at the final expression for the SNR gain using a matched-filter compared to a uniform acquisition:

$$\frac{\text{SNR}_{\text{matched}}}{\text{SNR}_{\text{uni}}} = \left( \frac{\sigma_r}{\sqrt{2 \pi} \text{erf}(k_{\text{max}} \sigma_r / \sqrt{2})} \right)^{\frac{1}{2}} \cdot \sqrt{2} \cdot k_{\text{max}} \cdot \text{erf}(k_{\text{max}} \sigma_r)$$

$$= \left( \frac{\sqrt{2} \cdot k_{\text{max}} \cdot \text{erf}(k_{\text{max}} \sigma_r)}{\text{erf}(k_{\text{max}} \sigma_r / \sqrt{2})} \right)^{\frac{1}{2}}.$$  

(A.4)

Appendix B

We derive the readout gradient time course $G(t)$ realizing a Gaussian acquisition density $d_{\text{acq}}(k)$ on a k-space traverse from $-k_{\text{max}}$ to $k_{\text{max}}$, as follows: By inserting the Gaussian target density of Eq. (A.1) into the differential Eq. (11), we first yield a concrete differential equation for the one-dimensional readout trajectory time course $k(t):=k_r(t)$:

$$|k| = \frac{1}{d_{\text{acq}}(k)} = \bar{C} \cdot \exp \left( -\frac{k^2 \sigma^2}{2} \right)$$

(B.1)

where tilded $C$ refers to a constant of no interest.
As \( k \) should increase monotonously during a traverse, \( k \geq 0 \) and we can neglect the absolute value in Eq. \((B.1)\). Following a logarithmic transform and differentiation, Eq. \((B.1)\) appears in the normal form of a second order non-linear ordinary differential equation

\[
\ln \left( k - \frac{1}{2} k^2 \right) = -\frac{1}{2} k - \sigma_T^2 f_k k = 0 \quad \Rightarrow k - \frac{1}{2} k^2 = k = 0. \tag{B.2}
\]

The general solution for this differential equation reads

\[
k(t) = \frac{\sqrt{2}}{\sigma_r} \cdot \text{erf}^{-1}(\frac{\sqrt{2}}{\pi} \sigma_r \sigma_{T,\text{avg}}(t + c_2)) \tag{B.3}
\]

with \( \text{erf}^{-1} \) being the inverse error function and \( c_{1,2} \) constants to be determined via side conditions.

The side conditions arise as the interval \(-k_{\text{max}} \) to \( k_{\text{max}} \) has to be covered within a traverse duration \( T_{\text{traverse}} \), i.e.

\[
k(0) = -k_{\text{max}} \Rightarrow -\frac{\sqrt{2}}{\pi} \sigma_r \sigma_{T,\text{avg}} \frac{\sqrt{2}}{2} \Rightarrow c_1 c_2 = \frac{T_{\text{traverse}}}{2} \tag{B.4}
\]

Dividing Eq. \((B.5)\) through Eq. \((B.4)\) and back-substitution into Eq. \((B.4)\) provides the values for \( c_{1,2} \) as

\[
-1 = \frac{T_{\text{traverse}}}{2} c_2 - c_2 = \frac{T_{\text{traverse}}}{2} \tag{B.6}
\]

From that, we yield the final form for the \( k \)-space trajectory evolution as

\[
k(t) = \frac{\sqrt{2}}{\sigma_r} \cdot \text{erf}^{-1}(\frac{\sqrt{2}}{\pi} \sigma_r \sigma_{T,\text{avg}} \frac{\sqrt{2}}{2} \frac{t}{T_{\text{traverse}}} - 1) \tag{B.8}
\]

Taking the temporal derivative of Eq. \((B.8)\), we yield the gradient waveform as specified in Eq. \((12)\):

\[
G(t) = \frac{\dot{k}(t)}{\gamma} = \frac{\sqrt{2} \pi}{\gamma T_{\text{traverse}} c_1} \exp(\text{erf}^{-1}(\frac{\sqrt{2}}{\pi} \sigma_r \sigma_{T,\text{avg}} \frac{\sqrt{2}}{2} \frac{t}{T_{\text{traverse}}}) - 1) \tag{B.9}
\]

References


