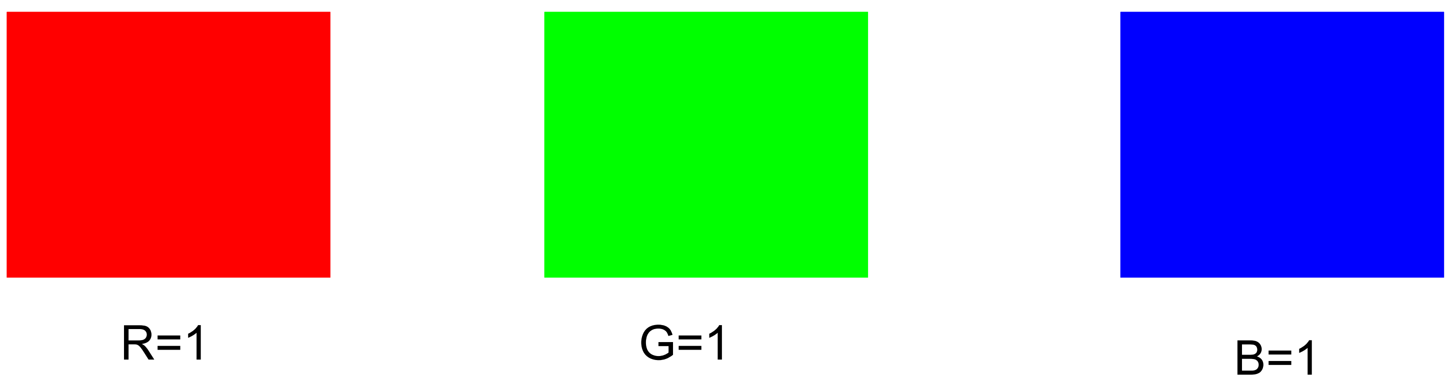
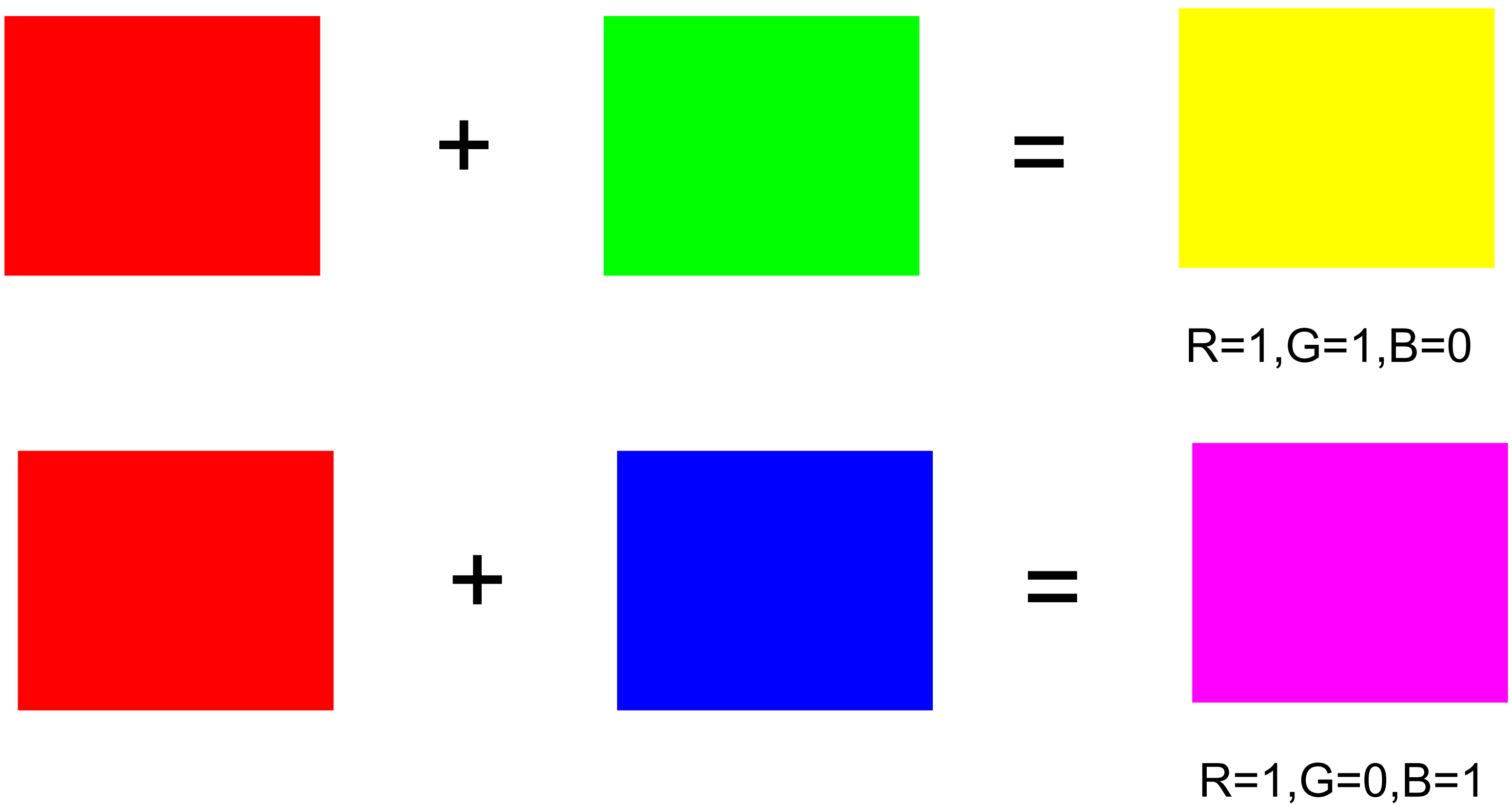
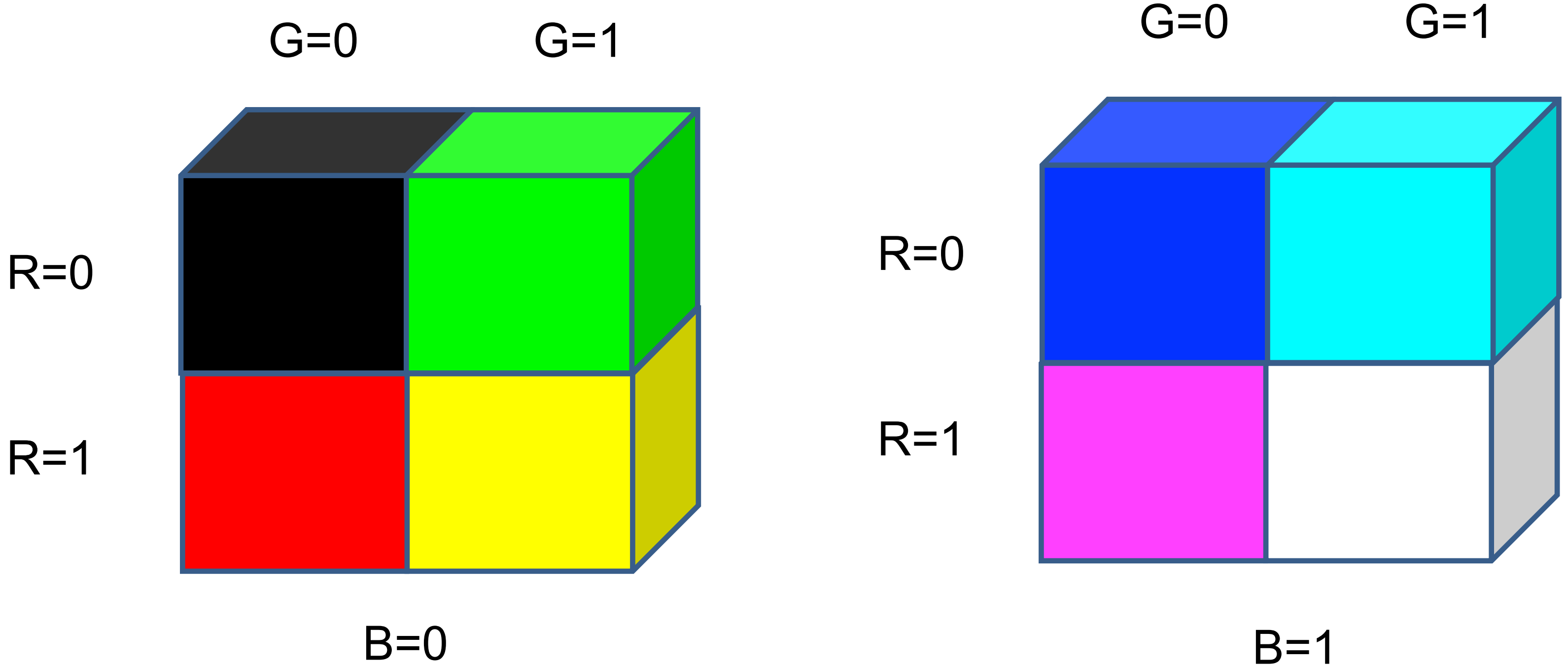
A model of colors



A model of colors

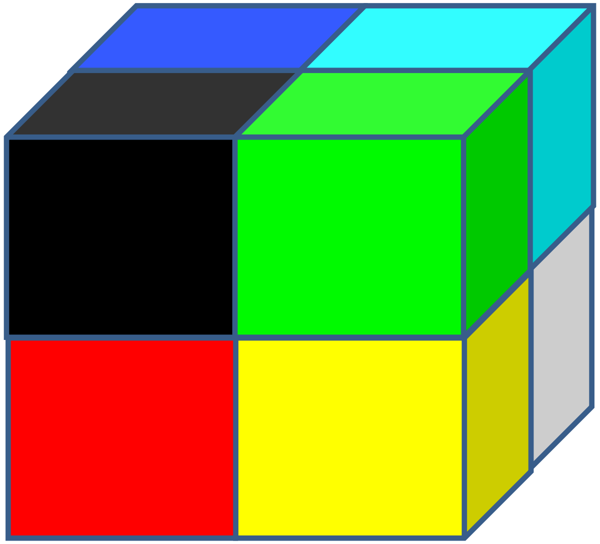


A model of colors



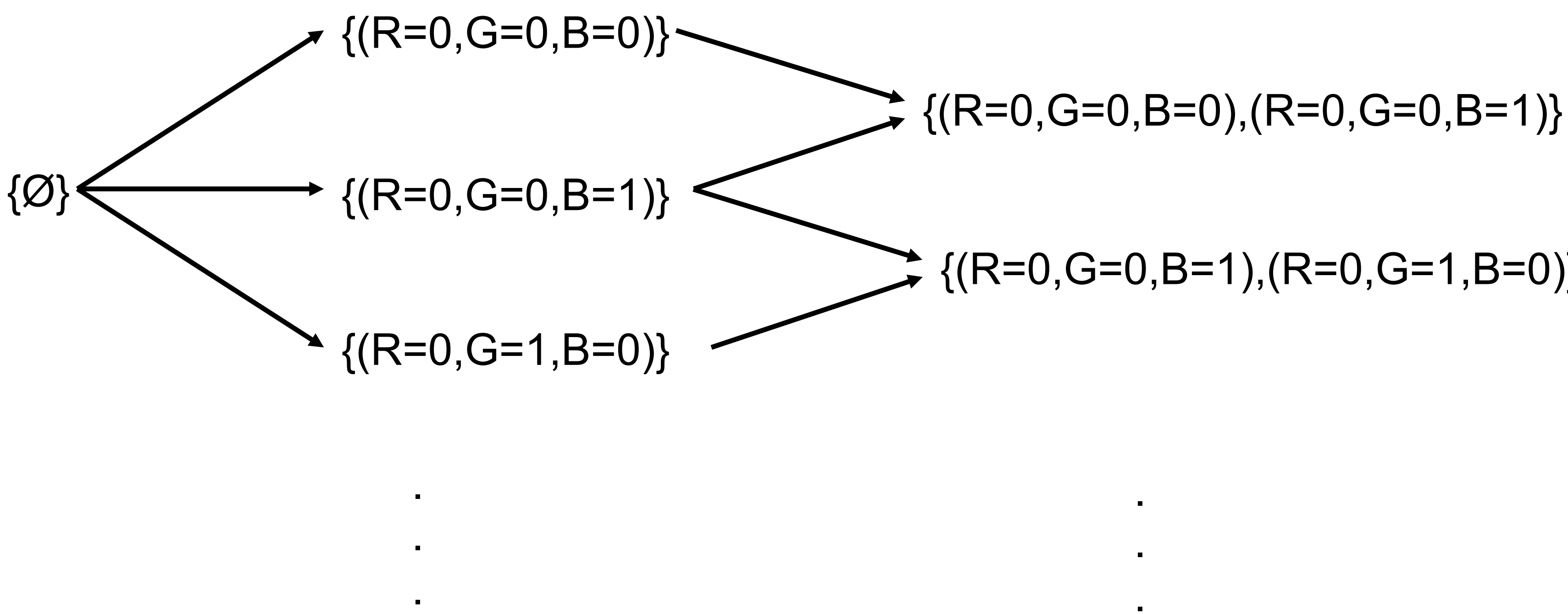
A model of colors

The outcome set is called Ω.
The cardinality of Ω is #(Ω) = 8.
We are interested in the power set P(Ω).



A model of colors

The power set is the set of all possible sets of Ω.



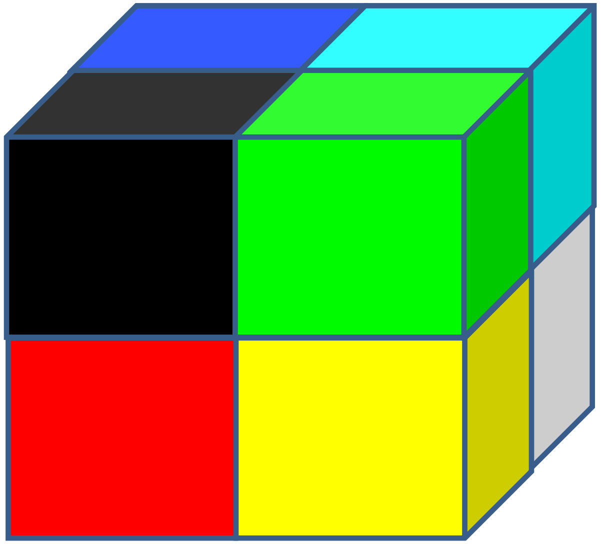
A model of colors

- We define a probability function F as the function with the following properties:
- $F : P(\Omega) \to [0,1]$,
- $F(\Omega) = 1$.
- If w and u are disjoint, then
  - $F(w \cup u) = F(u) + F(w)$.

A model of colors

Because of the additivity property, we need only to define the probability for each element in the outcome space. We call this function the joint probability.

$F(R = 0, G = 0, B = 0) = 0.432$
$F(R = 1, G = 0, B = 0) = 0.048$
$F(R = 0, G = 1, B = 0) = 0.288$
$F(R = 1, G = 1, B = 0) = 0.032$
$F(R = 0, G = 0, B = 1) = 0.006$
$F(R = 1, G = 0, B = 1) = 0.054$
$F(R = 0, G = 1, B = 1) = 0.014$
$F(R = 1, G = 1, B = 1) = 0.126$



A model of colors

Marginalization:

$$F(R,G) \ = \ F(R,G,B=0) \ + \ F(R,G,B=1)$$

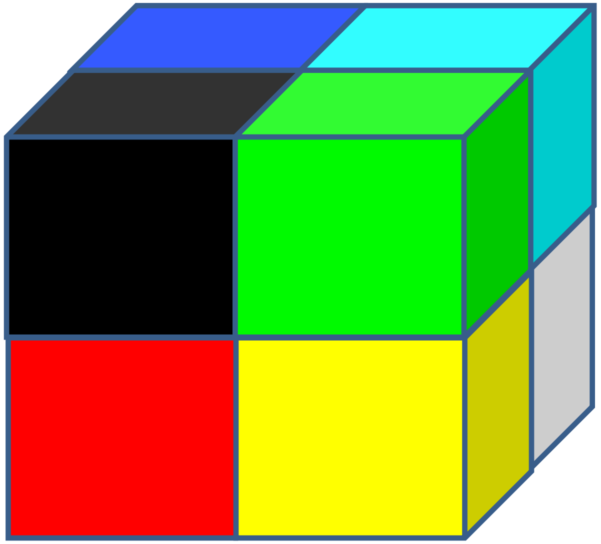
Conditional distribution:

$$F(R,G,B=0) = \ F(R,G|B=0)F(B=0)$$
$$F(R,G|B=0) = F(R,G,B=0)/F(B=0)$$

A model of colors

Exercise:
Compute:
F(B), F(R,G|B=0) and F(R,G|B=1)
What is the interpretation of each of these values in the cube?

$$F(R=0, G=0, B=0) = 0.432$$
$$F(R=1, G=0, B=0) = 0.048$$
$$F(R=0, G=1, B=0) = 0.288$$
$$F(R=1, G=1, B=0) = 0.032$$
$$F(R=0, G=0, B=1) = 0.006$$
$$F(R=1, G=0, B=1) = 0.054$$
$$F(R=0, G=1, B=1) = 0.014$$
$$F(R=1, G=1, B=1) = 0.126$$



Independent and conditionally independent variables

Given a picture of 1024x1024 pixels, in which each pixel has 8 possible values, how many cells does the hypercube has?

$$2^{3\times1024\times1024}$$

Independent and conditionally independent variables

Two variables are said to be independent if

$$F(R, G) = F(R)F(G)$$

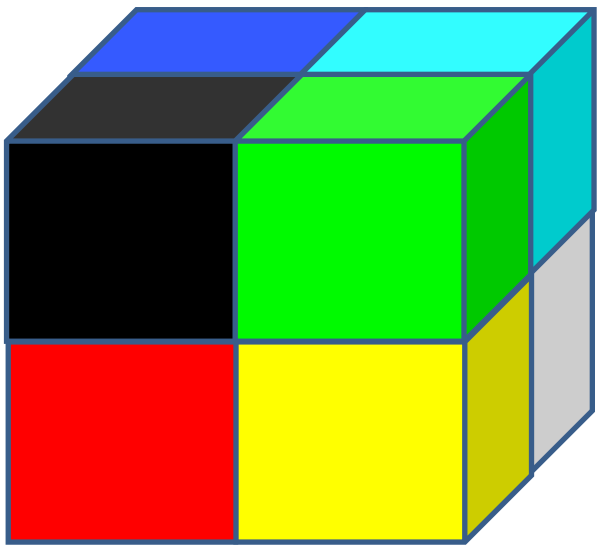
Two variables are said to be conditionally independent if

$$F(R|B)F(G|B) = F(R, G|B)$$

A model of colors

Exercise: Are R and G independent? Are they conditionally independent given B? What is the interpretation of independence in the cube?

$$F(R=0, G=0, B=0) = 0.432$$
$$F(R=1, G=0, B=0) = 0.048$$
$$F(R=0, G=1, B=0) = 0.288$$
$$F(R=1, G=1, B=0) = 0.032$$
$$F(R=0, G=0, B=1) = 0.006$$
$$F(R=1, G=0, B=1) = 0.054$$
$$F(R=0, G=1, B=1) = 0.014$$
$$F(R=1, G=1, B=1) = 0.126$$



General observations

- Conditioning can be understood as making an observation or performing an experiment.

- **Inference in the Bayesian sense is computing the conditional distributions of some random variables.**

- The floor is wet. What is the probability that it rained today?

- Donald Trump won the election. What is the probability that the Swiss Franc goes above 0.92 Euros?

General observations

The continuous case is technically more involved but very similar.

Mostly, one is provided not with the probability function F, but with the probability density function $dF(R < t)/dt \; = \; p(t)$. This is the derivative of the probability that R is lower than t.

In the following I will call the probability density function (pdf) of the random variable R the density or the probability of R=t.

Some more concepts

- Expected value of a function

$$\int p(R = t) f(t) dt = E[f(R)]$$

- Moments of a density

$$E[R^n]$$

- Mean (first moment)

$$E[R] = \bar{R}$$

- Variance:

$$Var[R] = E[(R - E[R])^2] = E[R^2] - E[R]^2$$

Basic concepts

- Moment generating function

$$\phi_R(w) = E[e^{wR}]$$

Note that

$$\frac{\partial \phi}{\partial w} = E[e^{wR} R]$$

$$\left.\frac{\partial^n \phi}{\partial w^n}\right|_{w=0} = E[R^n]$$

Transformation of random variables

- If the R follows the distribution
$$p(R = t) = p(t)$$
- Then $g(R)$ follows the distribution
$$p\big(g(t)\big)g'(t)$$
- This follows from the conservation of mass
$$\int_a^b p(t)dt = \int_{g^{-1}(a)}^{g^{-1}(b)} p\big(g(t)\big)g'(t)dt$$

The Gaussian distribution

- The standard normal Gaussian distribution

$$p(R = t) = \frac{1}{\sqrt{2\pi}} \exp - \frac{1}{2} t^2$$

This distribution is well defined as

$$\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2} t^2\right) dt = \sqrt{2\pi}.$$

The Gaussian distribution

- Exercise:
- If R is Gaussian distributed, what is the distribution of $\sigma R + \mu$
- What are the first two moments?
- Verify the result using the moment generating function

$$E[e^{wR}]$$

The Gamma distribution

- The Gamma function is

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} \exp -t \, dt$$

- For integer α:

$$\Gamma(\alpha) = \alpha!$$

- More generally

$$\Gamma(\alpha + 1) = \Gamma(\alpha)\alpha$$

- The Gamma distribution is defined as:

$$\Gamma(t; \alpha) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} \exp -t$$

The Gamma distribution

- Exercise:
- If R is Gamma distributed with parameter α, what is the distribution of $\beta \mathrm{R}$ for $\beta > 0$?
- Compute the mean and the variance of βR?