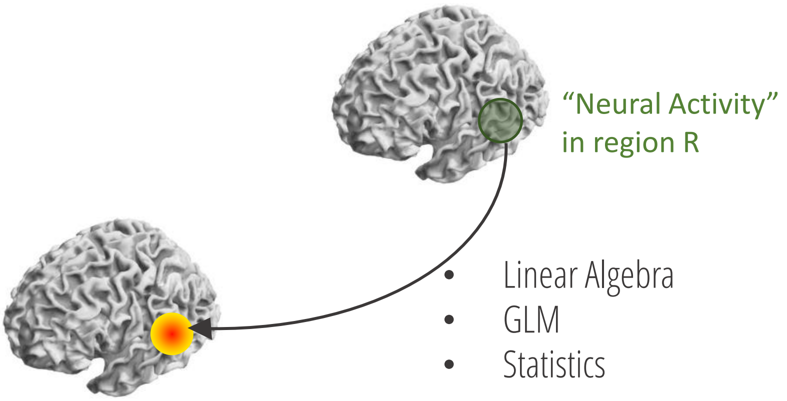
Cheat Sheet: From Linear Algebra to Significance



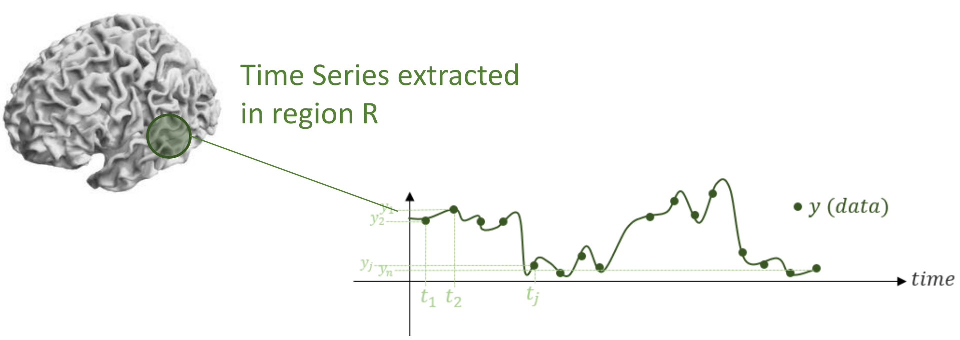
- Linear Algebra
- GLM
- Statistics

"...significant decrease in activation *($p < 0.05$)* in region R during presentation of ..."

Glossary:

β	Scalar Values
$\vec{x}$	Vector
X	Matrix
X^T	Matrix transpose
$\vec{\epsilon}$	i.i.d. noise vector
n, m	dimensions

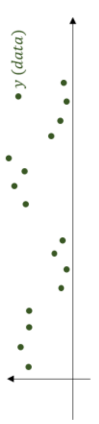
What fMRI data has to do with Linear Algebra



Note:

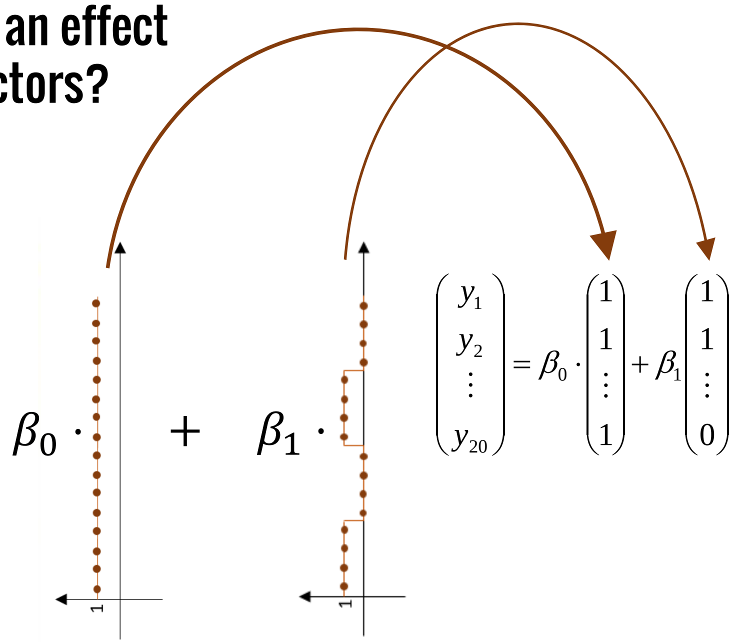
We don't really measure a continuous signal, but sample the data at discrete timepoints $t_1, t_2, \ldots, t_n$. Since at these times, we measure discrete signal values $y_1, y_2, \ldots, y_n$, we can nicely represent them in an n-dimensional vector $\vec{y}$.

How do we test for an effect of experimental factors?



Can we explain the signal as a linear combination of our regressors?

$=$



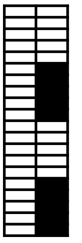
$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{20} \end{pmatrix} = \beta_0 \cdot \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} + \beta_1 \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_{20} \end{pmatrix}$$

Generalized Linear Model (GLM)

$$\vec{y} = \beta_0 \cdot \overrightarrow{x_0} + \beta_1 \cdot \overrightarrow{x_1} + \vec{\epsilon} = X \cdot \mathrm{B} + \vec{\epsilon}$$

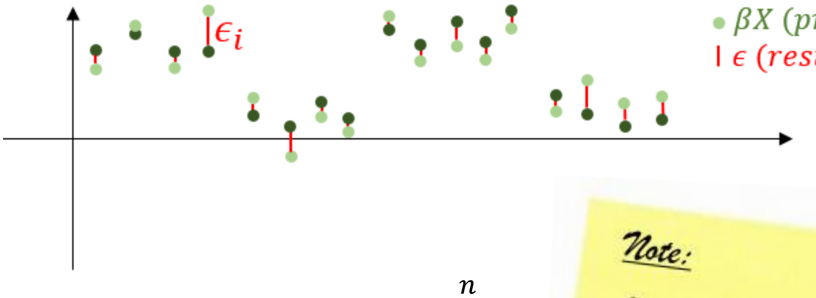
$$X = [\overrightarrow{x_o}, \overrightarrow{x_1}]$$ "Design Matrix"

$$X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 0 \\ 1 & 0 \end{pmatrix} =$$



$$\mathrm{B} = [\begin{matrix}\beta_0 \\ \beta_1\end{matrix}]$$

How to find the β-Values?



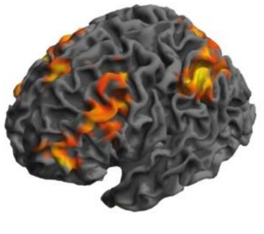
Minimize the error = explain the signal as good as possible! (*Least squared error*)

$$\hat{\beta} = argmin_\beta = ||\vec{y} - \beta X||^2 = \sum_{i=1}^{n} \epsilon_i^2$$

$$\rightarrow \hat{\beta} = (X^T X)^{-1} X^T \vec{y}$$

> **Note:**
> β-values are a measure of 'how much' of an experimental factor is present in the signal, i.e. how much of the signal variance is explained by a regressor.

Is the effect significant?



"…significant decrease in activation (*p < 0.05*) in region R during presentation of …"

Estimate

$$\widehat{\beta_i}$$

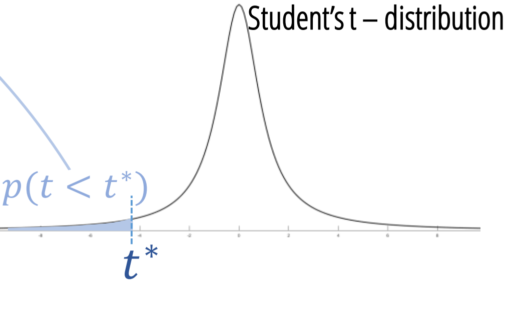
Standard error of estimate*

$$SEM_i = \frac{1}{\sqrt{\sigma^2 c_i^T (X^T X)^{-1} c_i}}$$

$$c_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \quad i^{th} \text{ entry}$$

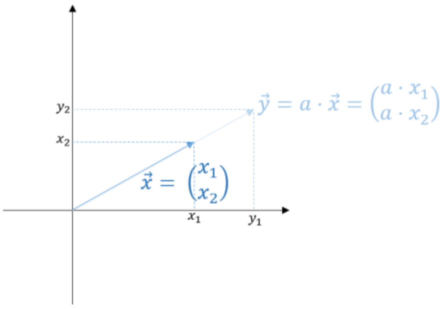
Test statistic

$$t^* = \frac{\widehat{\beta_i}}{SEM_i}$$



Basics of Linear Algebra

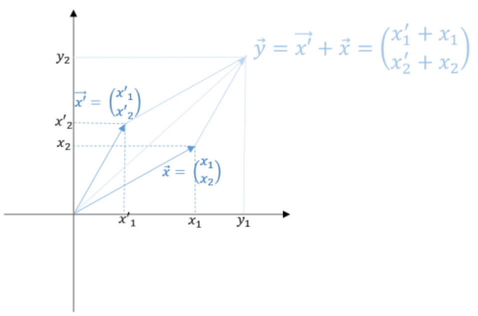
Multiplication with scalar $\boldsymbol{a} \in \mathbb{R}$



$$y_j = a \cdot x_j$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = a \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a \cdot x_1 \\ a \cdot x_2 \\ \vdots \\ a \cdot x_n \end{pmatrix}$$

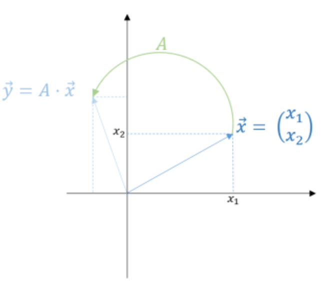
Vectoraddition $\overrightarrow{\boldsymbol{x'}} \in \mathbb{R}$

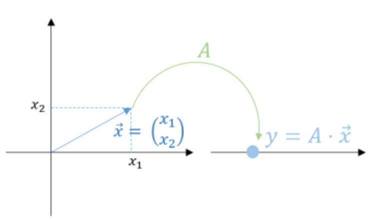


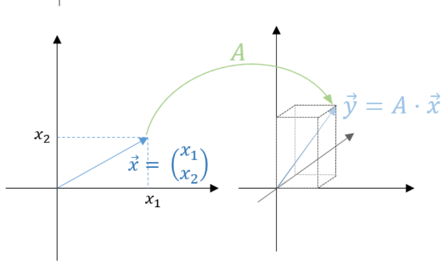
$$y_j = x'_j + x_j$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x'_1 + x_1 \\ x'_2 + x_2 \\ \vdots \\ x'_n + x_n \end{pmatrix}$$

Matrixmultiplication $\mathbf{A} \in \mathbb{R}^{\boldsymbol{m\,x\,n}}$







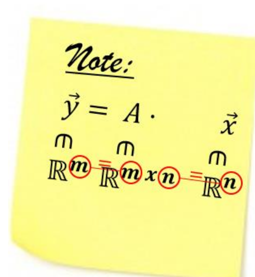
$$A: \mathbb{R}^{\boldsymbol{n}} \rightarrow \mathbb{R}^{\boldsymbol{m}}$$

m columns

n rows

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \bullet \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\vec{y} = \boldsymbol{A} \cdot \vec{\boldsymbol{x}}$$



$$y_j = \begin{matrix} a_{j1} \cdot x_1 + \\ a_{j2} \cdot x_2 + \\ \vdots \\ a_{jn} \cdot x_n \end{matrix} = (a_{j1} \cdot x_1 + a_{j2} \cdot x_2 + \ldots + a_{jn} \cdot x_n) = \sum_{k=1}^{n} a_{jk} \cdot x_k$$