Methods and Models 2017

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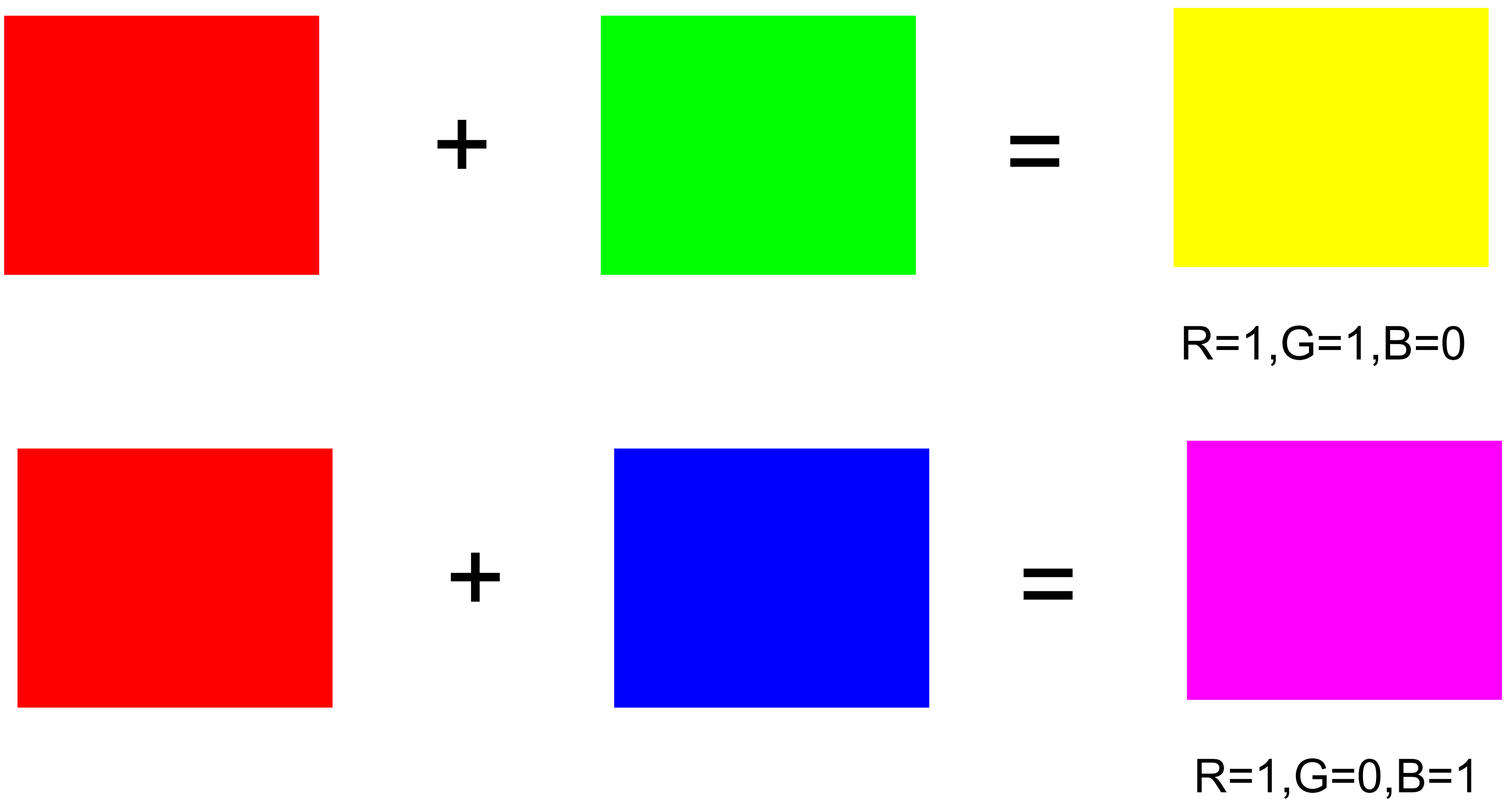
A model of colors

R=1 G=1 B=1

A model of colors



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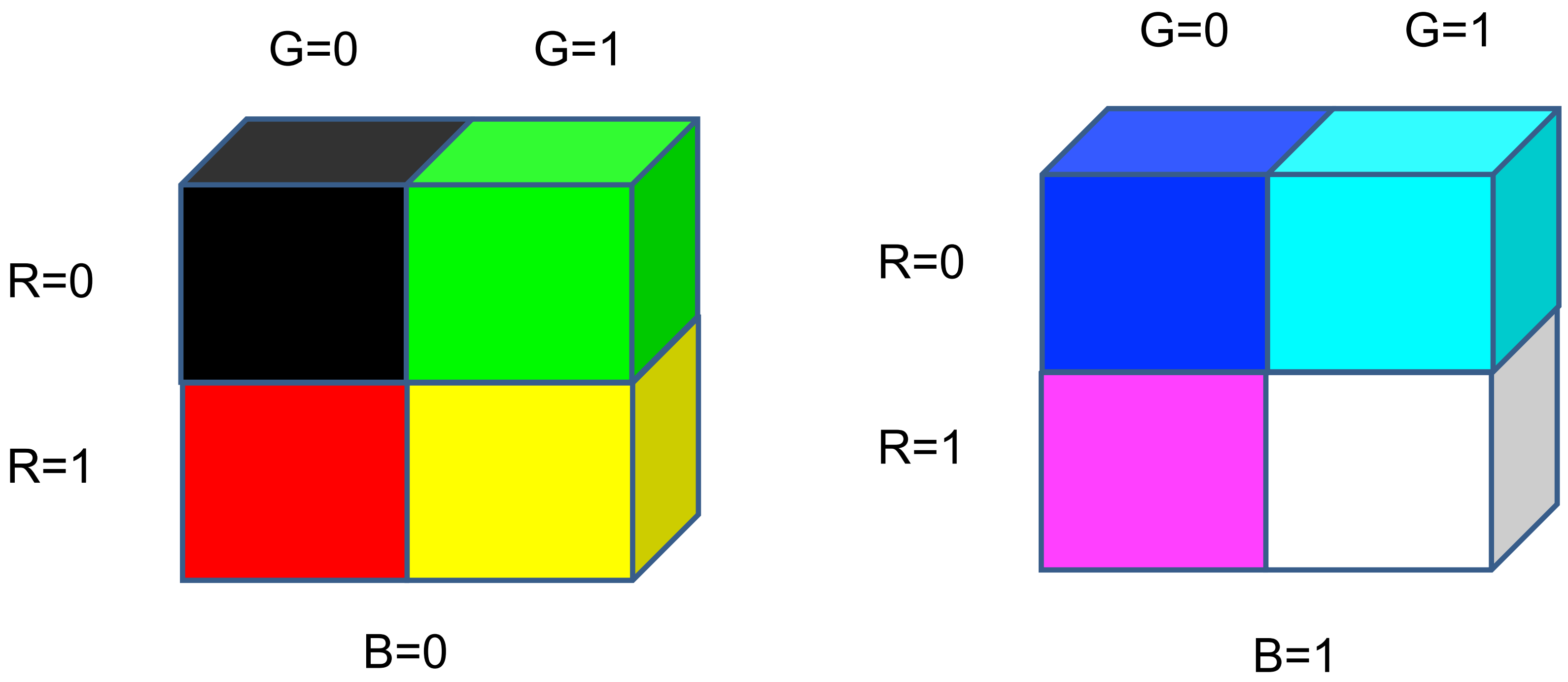
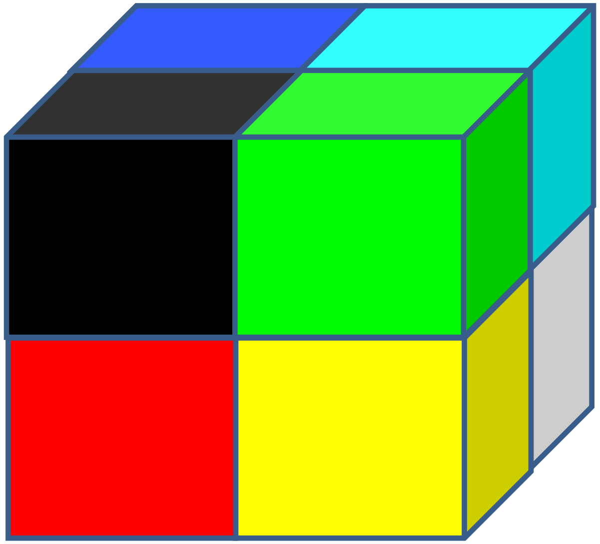


A model of colors

The outcome space is $R \times G \times B = \Omega$.
The cardinality of Ω is #(Ω) = 8.
We are interested in the power set P(Ω).



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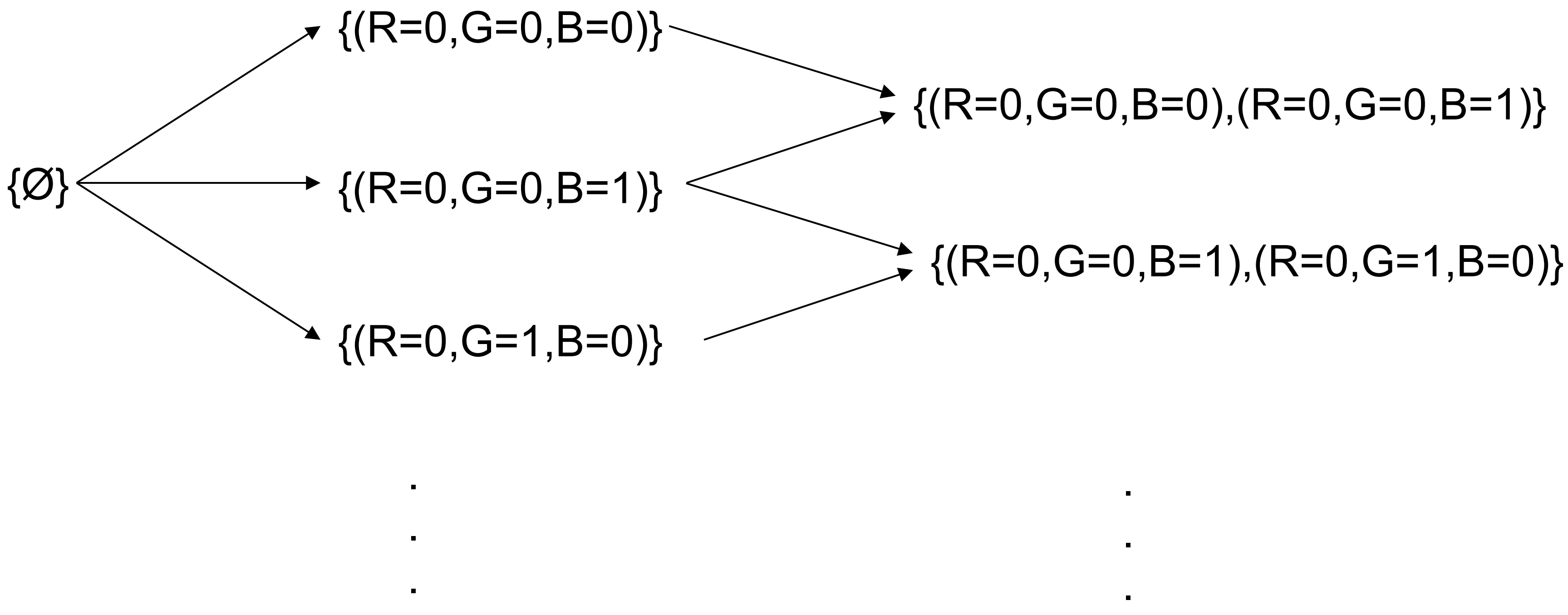
The power set is the set of all possible sets of Ω.



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We define a distribution F as a function with the following properties:

$$F : P(\Omega) \rightarrow [0,1],$$

$$F(\Omega) = 1.$$

If A and B are disjoint, then

- $F(A \cup B) = F(A) + F(B).$

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Some properties:

$$F(\Omega \cup \emptyset) = F(\Omega) + F(\emptyset) = 1 + F(\emptyset) = F(\Omega)$$
$$F(\emptyset) = 0.$$

Complement

$$F(A \cup A^c) = F(A) + F(A^c) = 1$$
$$F(A^c) = 1 - F(A)$$

Sub additivity:

$$F(A \cup B) \leq F(A) + F(B).$$

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Formally, a probability space is a triple

$$(\Omega, \Sigma, F)$$

Ω: Outcome space

Σ: Sigma algebra over Ω.

F: Distribution function over Σ.

A sigma algebra is a subset of $\mathrm{P}(\Omega)$, $\Omega \in \Sigma$, that is closed under complement, and countable many unions.

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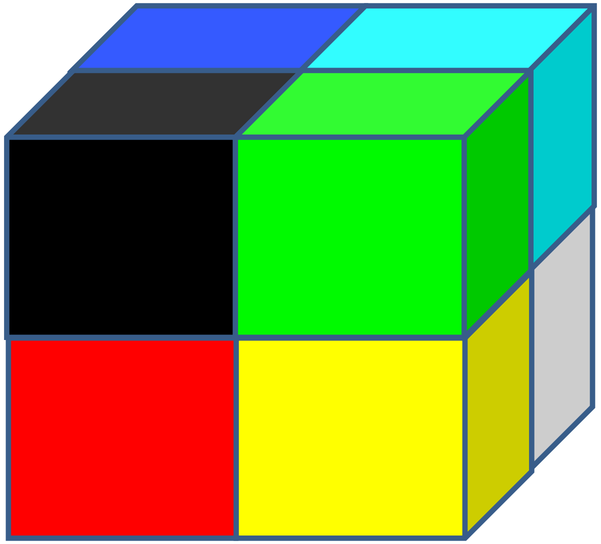
A sigma algebra is a subset of $\mathrm{P}(\Omega)$, $\Omega \in \Sigma$, that is closed under complement, and countable many unions.

We will never talk about this anymore.

A model of colors

Because of the additivity property, we only need to define the probability for each element in the outcome space. We call this function the joint probability.

$$F(R=0, G=0, B=0) = 0.432$$
$$F(R=1, G=0, B=0) = 0.048$$
$$F(R=0, G=1, B=0) = 0.288$$
$$F(R=1, G=1, B=0) = 0.032$$
$$F(R=0, G=0, B=1) = 0.006$$
$$F(R=1, G=0, B=1) = 0.054$$
$$F(R=0, G=1, B=1) = 0.014$$
$$F(R=1, G=1, B=1) = 0.126$$



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Marginalization:

$$F(R, G) = F(R, G, B = 0) + F(R, G, B = 1)$$

Conditional distribution:

$$F(R, G, B = 0) = F(R, G | B = 0) F(B = 0)$$

A model of colors

Exercise

Compute: F(B), F(R,G|B=0) and F(R,G|B=1)

What is the interpretation of each of these values in the cube?

$$F(R=0, G=0, B=0) \ = \ 0.432$$
$$F(R=1, G=0, B=0) \ = \ 0.048$$
$$F(R=0, G=1, B=0) \ = \ 0.288$$
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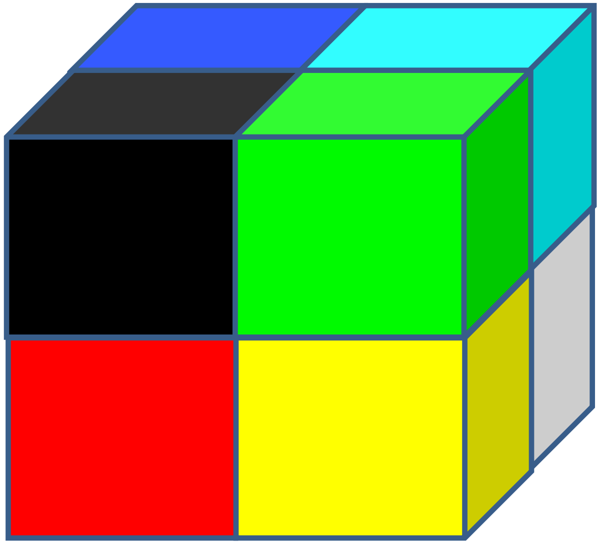

Independent and conditionally independent variables

Two variables are said to be independent if

$$F(R,G) = F(R)F(G)$$

Two variables are said to be conditionally independent if

$$F(R|B)F(G|B) = F(R,G|B)$$

A model of colors

Exercise: Are R and G independent? Are they conditionally independent given B? What is the interpretation of independence in the cube?

$$F(R=0, G=0, B=0) = 0.432$$
$$F(R=1, G=0, B=0) = 0.048$$
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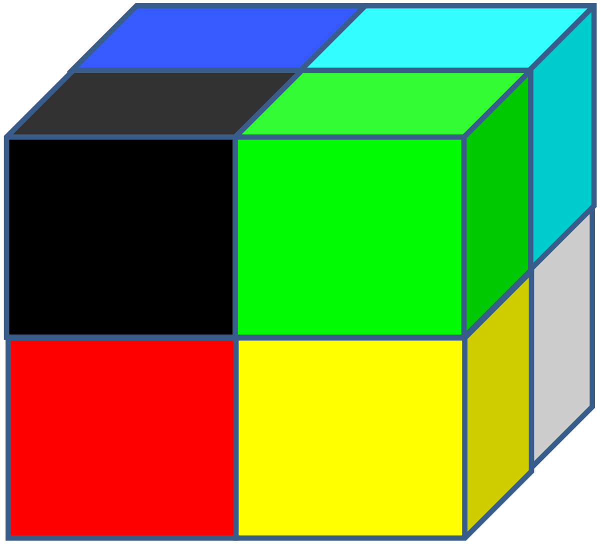


General observations

- **Inference in the Bayesian sense is computing the conditional distributions.**
- The floor is wet. What is the probability that it rained today?
- Donald Trump won the election. What is the probability that the Swiss Franc goes above 0.92 Euros?

Random variables

A random variable is defined as a (*measurable*) function from the outcome space Ω to $\mathbb{R}$ (*or any measurable space*).

It simply assigns a numerical value to each outcome.

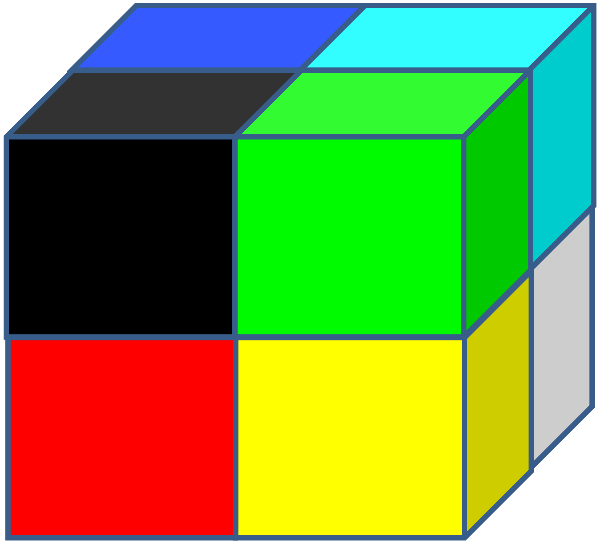
Example:

If you get the winning Lotto number you get 1'000.000CHF otherwise 0CHF.

Random variables

The random variable X assigns a numerical value to each element of the outcome space.

$$X(R = 0, G = 0, B = 0) \ = 0$$
$$X(R = 1, G = 0, B = 0) \ = 1$$
$$X(R = 0, G = 1, B = 0) \ = 1$$
$$X(R = 1, G = 1, B = 0) \ = 2$$
$$X(R = 0, G = 0, B = 1) \ = 1$$
$$X(R = 1, G = 0, B = 1) \ = 2$$
$$X(R = 0, G = 1, B = 1) \ = 2$$
$$X(R = 1, G = 1, B = 1) \ = 3$$

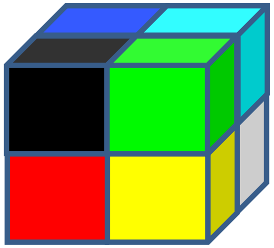


Random variables

The random variable X assigns a numerical value to each element of the outcome space.

We define the expected value of a random variable as

$$\sum_{\omega \in \Omega} X(\omega) F(\omega) = E[X].$$



Random variables

Exercise

We define G as: $G(\omega) = 1$ if $X(\omega) = 2$, and 0 otherwise.

$$X(R=0,G=0,B=0) = 0$$
$$X(R=1,G=0,B=0) = 1$$
$$X(R=0,G=1,B=0) = 1$$
$$X(R=1,G=1,B=0) = 2$$
$$X(R=0,G=0,B=1) = 1$$
$$X(R=1,G=0,B=1) = 2$$
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$$X(R=1,G=1,B=1) = 3$$

$$F(\{R=0,G=0,B=0\}) = 0.432$$
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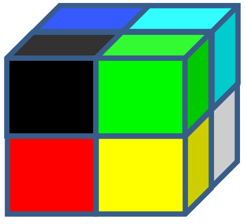
What is the expected value of X?
What is the expected value of G?



Random variables

The continuous case is technically more involved but similar.

Let [0,1] be the outcome space and

$$X: [0,1] \to [0,1],$$
$$X(t) = t.$$

For example, we define the distribution F as:

$$F[X(\omega) < t] = t.$$

Random variables

The continuous case is technically more involved but similar.

Example: The uniform distribution

$$\frac{dF[X(\omega) < t]}{dt} = p(t) = 1.$$

We say that X is standard uniformly distributed:

$$X \sim U_{[0,1]}$$

Random variables

More generally, the derivative dF/dt (if it exists) is usually called the **probability density function** and we will write it as

$$p(X = t) \text{ or } p_X(t) \text{ or } p(t).$$

Expected values are defined as:

$$E[X] = \int_\Omega X dF = \int_\Omega X(t) \frac{dF}{dt} dt.$$

Random variables

Following the previous example:

If dF/dt=1 and $X(t)$=t.

$$\int_{[0,1]} X dF = \int_{[0,1]} X(t) \frac{dF}{dt} dt = \int_0^1 t dt = \frac{1}{2} t^2 \Big|_0^1 = 0.5.$$

During the lecture you will encounter a lot expressions like:

$$E[X] = \int x \, p(X = x) dx = \int x \, p(x) dx$$

Some more concepts

Expected value of a function:

$$E[f(X)] = \int p(X = x) f(x) dx$$

Moments of a density

$$E[X^n]$$

Mean (first moment)

$$E[R] = \bar{R}$$

Variance:

$$Var[R] = E[(R - E[R])^2] = E[R^2] - E[R]^2$$

Basic concepts

Moment generating function

$$\phi_X(w) = E[e^{wX}]$$

Note that

$$\frac{\partial \phi}{\partial w} = E[e^{wX} X]$$

$$\left.\frac{\partial^n \phi}{\partial w^n}\right|_{w=0} = E[X^n]$$

Basic concepts

Exercise:

Compute the mean and variance of the uniform distribution using the moment generating function.

Remember l'Hospital rule:

If $\lim_{x \to 0} f(x), g(x) = 0, g(x) \neq 0$ for $x \neq 0$.

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)}$$

Uniform distribution: $F[X < t] = t,$

$$\phi_X(w) = E[e^{wX}] \;\; and \frac{\partial^n \phi_X(w)}{\partial w^n}\bigg|_{w=0} = E[X^n].$$

Transformation of random variables

Transformation of random variables

We want to find a function that preserves the measure:

$$F[a < X < t] = G[\phi^{-1}(a) < Y < \phi^{-1}(t)]$$

We denote the function

$$f(x) = F[a < X < x]$$
$$g(y) = G[\phi^{-1}(a) < Y < y]$$

with the property that

$$f(x) = g(\phi^{-1}(x))$$

and

$$\phi^{-1}(x) = y$$

The Gaussian distribution

- Exercise:
- If R is Gaussian distributed, what is the distribution of $\sigma R + \mu$
- What are the first two moments?
- Verify the result using the moment generating function

$$E[e^{wR}]$$

The Gamma distribution

- The Gamma function is

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} \exp -t \, dt$$

- For integer α:

$$\Gamma(\alpha) = \alpha!$$

- More generally

$$\Gamma(\alpha + 1) = \Gamma(\alpha)\alpha$$

- The Gamma distribution is defined as:

$$\Gamma(t; \alpha) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} \exp -t$$

The Gamma distribution

- Exercise:
- If R is Gamma distributed with parameter α, what is the distribution of $\beta \mathrm{R}$ for $\beta > 0$?
- Compute the mean and the variance of βR?